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AND

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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

REPORTS

WATER SUPPLY ENGINEERING[1]

THIRD REPORT OF THE COMMITTEE OF THE SANITARY ENGINEERING DIVISION

Since the publication of its First and Second Reports[2] the Committee of the Sanitary Engineering Division on Water Supply Engineering has continued to consider as its special field of activity the reporting of any noteworthy progress in the science or art of water supply engineering.

Materials and Methods of Construction

Cement and Concrete with Respect to the Prevention of High Setting Temperatures.—The Second Report contained descriptions of methods of preventing high temperatures in the mass concrete of dams by two methods:

(*a*) By control of the composition of the cement as in the Morris Dam for the Pasadena, Calif., water supply and the Boulder Canyon Dam; and

(*b*) The cooling of the concrete by refrigerating pipes distributed through it. The Committee has had further correspondence with the respective Chief Engineers of these projects with a view to recording authoritatively the experience to date with these methods.

It was found that comprehensive, up-to-date reports or monographs were already in print on all these developments. Among these monographs is one by R. W. Stenzel and S. B. Morris,[3] M. Am. Soc. C. E., giving the technical procedure used in testing the heat-producing properties of cement. A summary of experience with the concrete-cooling installation at Boulder Dam has been written by Bryan W. Steele, Assoc. M. Am. Soc. C. E.[4] It shows that the cooling pipes embedded in the dam have been successful in bringing each completed part to the temperature pre-determined as best for the harmonious action of the structure under stress. A further development is the elaborate system of radial and circumferential joint planes with batteries of grouting pipes leading to each joint, and copper strips dividing the joint planes into areas for separate grouting.

NOTE.—Written discussion on this Third Report will be transmitted directly to the Chairman of the Committee for possible use in preparing subsequent reports.

[1] Presented at the meeting of the Sanitary Engineering Division at New York, N. Y., on January 17, 1935.

[2] *Proceedings*, Am. Soc. C. E., December 1934, p. 1463 (two reports combined as Pts. 1 and 2, respectively).

[3] *Industrial and Engineering Chemistry*, July 15, 1934.

[4] *Engineering News-Record*, October 11, 1934.

Caisson Core-Walls for Dams

An unusual application of the caisson method to the construction of a deep core-wall in wet ground was made at the Quabbin Reservoir Dam to furnish additional water supply for the Boston, Mass., Metropolitan District. The project required the building of core-walls for dam and dike to depths as great as 130 ft. After two exploratory caissons were sunk, it was decided to build the core-walls in caissons placed end to end. Pumping equipment which lowered the ground-water level reduced the air pressure necessary to operate inside the caissons. Thirty-one caissons required for the core-wall of the dike have been sunk and sealed to the ledge rock.

Extra Large Coarse Aggregate in Concrete of Dams

It has become common practice in recent years to use stones as large as 16 in. in diameter in the coarse aggregate of concrete for dams.

Pumping Concrete

A special apparatus has been introduced to pump plastic concrete through pipe for distances at least as great as 850 ft. Caleb M. Saville, M. Am. Soc. C. E., has supplied the Committee with details of the use of this new device in placing concrete in the Bills Brook Dam of the Metropolitan Water District supplying Hartford, Conn., and adjacent municipalities. Concrete is proportioned by weight, mixed for 1½ min in a 1-cu yd concrete mixer and delivered through a short inclined chute to a 2-cu yd remixing hopper equipped with a series of vertically mounted paddles which revolve slowly and prevent segregation of aggregates. The remixing hopper is connected through a circular flanged opening in its base to the conical intake section of the "pump."[5]

The following statistics relate to the Bills Brook Dam work to date:

Total concrete placed, in cubic yards		26 000
Average rate of pumping, in cubic yards per hour	about	25
Maximum rate of pumping, in cubic yards per hour	about	33
Maximum sustained rate of pumping, in cubic yards per hour	about	28
Longest distance pumped, in linear feet (750 ft of straight pipe plus four 45° and two 22½° bends)		850
Maximum elevation above mixer, in feet		20
Maximum distance below mixer, in feet		50

Earth Dams

A new system of controlling the moisture content and of compacting the embankment material has been used with apparent success in constructing three dams for the Los Angeles (Calif.) Water Supply. Under the system, optimum moisture content of available embankment materials for a given intensity of compacting pressure may be measured by a simple newly devised apparatus, and embankments designed for requisite stability and water-tightness. Its use has demonstrated the great beneficial effect of proper compacting in increasing impermeability.

[5] For a more complete description, see "Pumping Concrete for Dam Construction," by E. W. Sherman, Assoc. M. Am. Soc. C. E., *Civil Engineering*, August, 1934, p. 398.

New conceptions, developed in part from principles of soil mechanics advanced by Charles Terzaghi, M. Am. Soc. C. E., appear to have been proved by extensive laboratory experiments with many kinds of soil. Thus, an embankment rolled by heavy equipment, but with deficient moisture, may make a hard embankment but may contain no more material and may be no more impervious than one made of the same material but with more moisture and rolled with lighter equipment. Furthermore, the hard embankment, in time, will soften to the same point as the other embankment having the same dry material content; and a dam of uniform and sufficiently impervious material is as safe as one that has the conventional arrangement (that is, with porous material in the down-stream part).

Other new conceptions affecting accepted principles of dam design have been described[6] by R. R. Proctor, Assoc. M. Am. Soc. C. E.

Cement Mortar Lining for Pipe Lines in Situ

A process for lining pipes in the ground with cement mortar, introduced by Mr. William T. Tate, of Australia, is being used extensively in New Zealand and Australia, and was used during 1934 in Manchester, England, for lining approximately 10 miles of 3-in., 4-in., 5-in., and 6-in. pipe. The pipe is first cleaned by the usual methods. A cable is placed in the pipe; a quantity of cement mortar is then injected at one end of the section of about 400 ft to be lined at one time, and a specially designed dehydrating mandrel is pulled through the pipe at the rate of about 20 ft per min. The Williams-Hazen coefficient of the resulting lining has been measured as from about 140 to 150 when based on the actual diameter of the lining, and 65 for 3-in. pipe to 100 for 5-in. pipe when based on the nominal diameter. The English linings are made about $\frac{3}{8}$ in. thick; they occupy a considerable part of the waterway and, consequently, reduce the coefficient based on nominal diameter. A lining $\frac{1}{8}$-in. thick has been applied successfully, and the lining has also been applied successfully to pipe lines 8 in., 9 in., and 12 in. in diameter laid in the Manchester (England) Pipe Yard for experimental purposes.

The foregoing information is from the personal observation, in England, of a member of the Committee.

Measurement of Tastes and Odors in Water

Notable progress has been made in the measurement of tastes and odors in water. The earlier work toward these improvements is to be credited to Charles H. Spaulding, Chemist of the Springfield (Ill.) Water-Works, and the later developments to John R. Baylis, Assoc. M. Am. Soc. C. E., and Gordon M. Fair, M. Am. Soc. C. E.

The original principle involved is that of dilution with odor-free water to the point where the odor is no longer perceptible. The intensity of odor is then designated by the multiple which the quantity of dilution water is of the quantity of water to be tested.

The more recent developments are in apparatus and technique which take into account the physical nature of the human olfactory organs. This entire

[6] *Engineering News-Record*, August 31, 1933, p. 245.

new system of odor measurement has found a place in the regular routine of many water supply laboratories, and supplements the "standard" method which is still in use.

Elimination of Tastes and Odor in Water Supply

Mr. Baylis reports rapid progress along the lines indicated in the First Report, namely, in the use of activated carbon. Powdered activated carbon is now used in more than 600 water-treatment plants in America, at times when the water has noticeable taste or odor. At some plants, carbon is used continuously, and the number of such plants is increasing. Thus far, the taste and odor from certain waste liquors from beet-sugar refineries have been found to be not removable by any quantities of carbon that may be used. George D. Norcom, Assoc. M. Am. Soc. C. E., notes that experience with surface water in areas suffering from extreme drought has shown that when a water becomes bad enough all methods now in use may fail to make it palatable.

Sterilization of Water

Liquid chlorine continues to be the most important agent for the sterilization of water, and improvements have continued to be made in the apparatus for its application. The use of ton containers for chlorine, formerly confined to large plants, is increasing owing to the economy. Mr. Norcom states that ammonic-chlorine (chloramine), at first heralded as a general taste-removing application, has come to be regarded largely as a means of carrying larger residuals of chlorine without causing the taste of chlorine. As the sterilizing power of chlorine is reduced by the addition of ammonia, it has become the policy to carry double the residual that would be required with chlorine alone. Ammonia, applied previous to chlorination to water containing small quantities of phenol or certain allied chemicals, will prevent the formation of taste-producing compounds which otherwise would be formed by the action of chlorine on these chemicals.

The perfection of readily soluble calcium hypochlorites has re-awakened interest in hypochlorites for the sterilization of water and for other filter-plant uses. The new hypochlorites are quite stable both dry and in solution; they do not clog orifices quickly; and they contain twice as much chlorine as the best of the older bleaching powders.

Effect of Weeds on Taste of Water and Control of Weed Growth

Original investigations regarding the effect of different species of weeds on tastes in water and the methods of controlling the growth of such weeds in reservoirs and canals have been made by G. E. Arnold, Assoc. M. Am. Soc. C. E., for the City of San Francisco, Calif.[7]

Each type of weed was studied separately in a tub of circulating water. (By "weed" is meant a rooted plant; not algæ.) Lily pads, *potamogeton, myriophyllum, cladophora,* and *lemna* (duck weed) were found to give no taste, whereas *chara* does give a taste.

[7] In an unpublished paper presented before the California Section of the Am. Water Works Assoc., at Long Beach, Calif., on October 24–27, 1934.

Copper sulfate, aeration, chlorination, chlorine, and ammonia were all tried unsuccessfully to remove the taste, while filtration through activated charcoal was very effective as long as the water was clear; if the water was turbid, filtration through sand and then charcoal was effective.

The best means of controlling weeds was by covering the reservoir to exclude sunlight. Some success was obtained by the use of a clay blanket. Growth was reduced by concrete lining, especially on areas of concrete covered by asphalt or tar. Growths have been killed by dropping the water levels in reservoirs temporarily, and have not resumed growing. Continuous feeding of copper sulfate was effective only if the flow was sufficient to change the water about every 24 hr; intermittent doses of copper sulfate were not effective.

Coagulants for Water Purification, and Their Application

Although alum continues to be the most valuable and most generally used coagulant, ferric chloride and ferric sulfate have been improved in form and price and are being promoted in competition with alum. The use of ferric sulfate at Providence, R. I., has been described by E. L. Bean, Chemist in charge, and L. H. Enslow, Assoc. M. Am. Soc. C. E., advises that chlorinated copperas is being used as the coagulant at Dallas, Tex., and Richmond, Va. The same authority states that at Oklahoma City, Okla., ferrous chloride, prepared at the water purification plant by the action of chlorine on iron scrap, is used successfully for coagulation.

Prevention of Corrosion in Pipes by Treatment of Water

The practice is growing more general of treating filtered water, also unfiltered waters which are naturally corrosive, with lime or, in a few cases, soda ash or caustic soda, to prevent or reduce corrosion and incrustation of mains and services. The process is controlled largely by the so-called Marble test and tests for the hydrogen-ion concentration, pH, and free carbon dioxide, CO_2. Such treatment, of course, cannot correct the loss of carrying capacity already suffered by previous incrustation. A permanent remedy for such loss of carrying capacity in mains 3 to 16 in. in diameter has been devised (see heading "Cement Mortar Lining for Pipe Lines in Situ"). Processes for re-coating large pipes which may be entered, were noted in the Second Report of the Committee.[8]

Frozen Pipes

The Committee wishes to make a brief record of the freezing of many pipes and services in water supplies in New England, New York, and the Middle West during and directly after some extraordinarily cold weather in the winter of 1933–34, following a long period of weather of more than usual winter severity. So various were the character and wetness of soil, depths of cover, climatic conditions, and degree of water circulation in the frozen pipes that few generalizations are possible as to needed cover. Depths of cover 3.5 ft on Long Island, New York, and 5 ft in Northern New York State, appear to be sufficient for mains and services.

[8] *Proceedings*, Am. Soc. C. E., December, 1934, p. 1478.

A number of salient points, mostly not new, were emphasized by the experience:

(1) Pipes freeze at least as frequently directly after the extreme cold has moderated as during the extreme cold. This appears to be explained adequately by the continued escape of heat from the pipe upward into the layers of ground which have not only become frozen by the extreme cold, but retain a temperature much below the freezing point.

(2) Service goose-necks from taps in tops of mains are the most vulnerable to freezing. Tapping on the sides of mains, with goose-necks kept as low as the centers of mains, have become, increasingly, the approved method.

(3) The practice of lowering the grade of a street without lowering the water mains—often without notifying the Water Department—is a frequent source of trouble. (No doubt neglect on the part of the Water Department may be partly to blame.)

(4) The cost of lowering mains and services to avoid a repetition of the experience of the winter of 1933–34 seems generally to be too great to permit a general lowering of all frozen services. In many places the art of thawing by electricity, and the consequent lowered cost of correcting the freezing trouble in this way, has reached a high state of development. Doubtless this has contributed greatly to the view of many managers that it is cheaper to pay a big bill for thawing once every 15 to 30 yr than to dig up and lower the pipes. Inability to secure cash for all needed improvements during this period of stringency is another important element in the decision.

(5) Even the best triple-wood coverings are not proof against the freezing and bursting of elevated tank risers which consist of pipe. Large steel risers do withstand the cold successfully. Overflow pipes within tanks are likely to be broken by ice in such extreme weather.

Hydraulics

Water-Hammer Problems in Water Supply Engineering.—The following statement is condensed from an article kindly furnished to the Committee by S. Logan Kerr, Assoc. M. Am. Soc. C. E.[9]

Although the earliest inquiries into water-hammer doubtless resulted from adverse experience in water supply systems, nearly all the recent studies on the subject have been made in connection with hydro-electric penstocks and supply lines.

Most modern theories on water hammer may be traced to the classic work of Joukovsky, in Moscow, in 1897, and Allievi's "Theory of Water Hammer," that appeared in Italy in 1903. The latter is based on three basic factors: (1) The nature of the pipe line (that is, size, thickness, and material); (2) the velocity of flow; and (3) the rate of change in velocity.

The maximum water-hammer is produced in a pipe line without branches when a gate on the line is shut in the short time required for the surge to travel from the gate to the source of the flow (open reservoir or tank) and return. The velocity of the surge in a municipal distribution system will

[9] *Water Works Engineering*, February 6, 1935.

vary between 3 600 and 4 200 ft per sec. This maximum water-hammer will be about 50 lb per sq in. multiplied by the change in velocity. Thus, a conduit 5 miles long with a velocity of flow of 3 ft per sec. would have a water-hammer of 150 lb per sq in. if the valve were closed in a time of $\frac{2 \times 5 \times 5\,280}{3\,600}$, or 15 sec.

In the past, the safety of the water supply distribution system has been due to the generally low velocity of flow, the slow closing of gates, and the use of steam pumps. Modern automatic valves and check-valves have accentuated the danger; moreover, with the introduction of the electric, motor-driven, centrifugal pump many cases of damage are recorded where the power supply to the motors failed and the water column in the discharge pipe was suddenly reversed and slammed the swing check-valve upon its seat. The same problem has been encountered in modern automatic check-valves and no panacea has as yet been produced.

In recent studies the theory of water-hammer has been developed to cover the more complex cases where pipe systems have different diameters, thickness, and material in different parts and where branch lines exist. The computations, while laborious, have been simplified as to form and may sometimes be aided by the use of a mechanical integrator. Simple "short-cut" solutions are sometimes sufficient for practical purposes.

Friction Coefficients in Distribution Systems

A Committee of the New England Water Works Association is collecting data on loss of head in pipes in service in various water systems. The First Report of the Committee stated that data from fifty representative cities showed large variations of the value of C in the Williams-Hazen formula for pipes of the same service age. Curves drawn to compare the value of C as predicted in the Williams-Hazen tables and actually observed values on unlined pipes show a marked difference. Capacity losses expected after 30 yr of service had been reached on the average after 13 yr of service.

Friction Loss in Bends

David L. Yarnell and the late Floyd A. Nagler, Members Am. Soc. C. E., have presented the more outstanding results of a series of experiments of the flow of water around 6-in. pipe bends of various shapes and degrees of curvature.[10] A novel feature of the investigation was the use of transparent material for the testing channels. The experiments show that:

(1) It is possible to have conditions in which resistance to flow may be very small or unusually large in the same pipe bend carrying identical quantities of water;

(2) In a standard 90°, 6-in. pipe bend for the same quantity of flow, with high velocity on the inside and low velocity on the outside of the approach pipe, the loss of head may be four times as much as would be measured in

[10] *Proceedings*, Am. Soc. C. E., March, 1934, p. 783.

the same bend when high velocity exists on the outside, and low velocity on the inside, of the tangent leading to the bend;

(3) Present formulas for computing loss of head due to bends appear to apply only to cases in which approximately uniform velocity distribution exists in the approach pipe;

(4) The losses of head in the bends experimented upon appear to vary as the square of the velocity and not the 2.25 power as suggested by some writers;

(5) A pipe bend may be as useful as any other device for the measurement of discharge;

(6) The direction of flow of the secondary currents in pipe bends depends entirely upon the velocity of distribution in the approach pipe; and

(7) The same fundamental laws of flow through bends apply to both closed conduits and open channels.

Progress of Committees

American Water Works Association Committee on Water Works Practice.—The text on "Quality and Treatment of Water" for the next edition of the Manual of Water Works Practice was finished, except the final editing, during 1934, and will be published in 1935. A new Tentative Standard for Water Works Gate Valves has been drafted and, with a 20-page report of investigations, has been distributed for criticism.

A Tentative Standard for Steel Stand-Pipes and Elevated Tanks previously published in the *Journal* of the Association, is undergoing slight revision before adoption as a final standard. Following the 48-page report on "Cross Connections Between Public Water Supplies and Industrial and Fire Supplies," published in the *Journal* of the Association in 1933, a supplementary report was issued, in 1934, especially on outbreaks in 1933 of amœbic dysentery.

Tentative Standards for Purchase of Chemicals Used in Water Purification Plants were presented to the Association in a report of more than 200 pages. This report is being edited for publication in 1935.

American Standards Association Committee A-21 on Cast-Iron Pipes.—This Committee has practically completed its extensive program of tests, including separate and combined bursting and crushing tests of pipe supported in the several common methods; tests of bursting strength of fittings; tests of loss of hydraulic head in fittings; tests of corrosion of various kinds of cast iron; tests of pipe linings and coatings, etc. The Committee is now engaged in applying the results of its experiments and researches in the design of schedules of pipes and fittings, and in the drawing of revised specifications. The Committee's experimental results and conclusions will be made available at a considerable number of libraries when the work has been completed.

Acknowledgments

In the list of associations connected with water supply engineering, which was given in the First Report, at least two additional names should have been included, namely, the Conference of State Sanitary Engineers and

the National Geophysical Union. Another year the Committee will endeavor to report what these associations have done, which should be added to the record. It is understood that the National Geophysical Union supplies convenient meterological, run-off, and silt data.

The Committee wishes to acknowledge the assistance received from a considerable number of engineers not connected with the Committee. The response has been ready and generous.

Respectfully submitted

T. H. WIGGIN, *Chairman*,
F. A. BARBOUR,
JOHN R. BAYLIS,
L. R. HOWSON,
F. H. STEPHENSON,
J. A. WADE,

Committee of Sanitary Engineering Division on Water Supply Engineering.

January 8, 1935.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

THE ADJUSTMENT OF A LEVEL NET

By George H. Dell,[1] Assoc. M. Am. Soc. C. E.

Synopsis

Two closely related methods of obtaining a "least squares" adjustment of the closing errors in a level net are described in this paper. The first method involves the writing, by inspection, of the appropriate normal equations and their subsequent solution by a process of converging increments; the second utilizes the principle of successive distributions, and is suggested by the Cross method of moment distribution.[2]

Both methods are well adapted to the use of the slide-rule and possess the advantages of simplicity and brevity, as compared with the conventional least squares solutions. They are intended to apply primarily to surveys of ordinary extent and accuracy, such as those executed by municipal and highway engineering organizations.

Beyond the consideration that, in practice, one method may appeal more than the other to a given individual, the reasons for presenting a dual treatment of the subject include the following: (*a*) The first method derives immediately from the principal hypothesis of the theory of least squares, and thus serves to substantiate the correctness of the second method; (*b*) the process utilized (in the first method) for solving the normal equations may be successfully applied to various other types of engineering problems (for example, to indeterminate structures); and (*c*) the parallel solutions show the inter-relations of the two methods and suggest a procedure that may be used to advantage in devising extensions of the distribution method to new types of problems.

Introduction

The precision that characterizes the planning and execution of present-day structures demands, in general, that reliable bench-marks be established

Note.—Discussion on this paper will be closed in August, 1935, *Proceedings*.

[1] Instr., Civ. Eng., Univ. of Illinois, Urbana, Ill.

[2] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

and maintained throughout, at least, the period of construction. In towns and cities, bench-marks should be created with a view to permanence.

The value of a bench-mark depends as much upon the reliability of the elevation assigned to it as upon its ability to maintain itself at a fixed height. Much difficulty is encountered in attempting to arrive at the correct elevation of bench-marks by field work alone, due to unavoidable discrepancies which are found when closing in from various lines. It appears, at times, that the more one levels in a given locality the greater is the uncertainty in the bench-marks.

Much of this difficulty in the field can be avoided by adopting, in part, the methods that have been used by the United States Coast and Geodetic Survey in the execution of many thousands of miles of precise levels. Particularly commendable in this connection are the establishment of a definite standard of precision and the practice of running all level lines twice for the purpose of putting them to test; in case of satisfactory agreement, the mean of the observed differences in elevation is then used.

In spite of all precautions, however, discrepancies still persist, and some form of adjustment is necessary. Generally speaking, there are three methods of adjusting elevations, namely, (1) by "guessing"; (2) by using a "rule of thumb" (which may be worse than guessing); and (3) by calculating the most probable values in accordance with the laws of probability.

The latter method, under the title of "least squares," has been in use for more than a century, but its application has been restricted rather closely to problems in astronomy and geodesy. For such purposes, a high degree of precision is generally required, and, consequently, the treatises dealing with that method are likely to appeal to the layman as being too elaborate for his requirements, and the calculations overly precise and laborious.

The aim of this paper is to present methods whereby the closing errors in a level net can be adjusted with approximately the same time and effort as are required to solve the somewhat analogous problem of finding the bending moments in a framed structure. These methods are proposed for use principally in connection with work done with the ordinary engineer's level, in which instance it is doubtful whether the precision obtainable in the field warrants an attempt to deal with elevations or readings in units of less than 0.01 ft, or 0.005 ft. Nevertheless, with a little added effort, the precision of the calculations is capable of further extension.

The use of a "polyphase" type of slide-rule is of great assistance in performing the necessary calculations, as in both methods an entire series of values can then be read off with a single setting of the slide.

Definitions and Preliminary Explanations

To facilitate description, the following terms will be used:

Control Bench-Mark: A bench-mark with an elevation which is to remain fixed, and to which the elevations of other bench-marks are to be adjusted.

Junction Point: A bench-mark at which three or more level lines meet.

Section: The length of level line connecting consecutive junction points or control bench-marks.

Intermediate Bench-Mark: A bench-mark located at some intermediate point of a section.

Sub-Section: A part of a section terminating at an intermediate bench-mark.

Circuit: A polygon or closed figure bounded by sections (but containing none within its interior).

Exterior Section: A section situated along the boundary of the net, and hence belonging to only one circuit.

Interior Section: A section situated within the interior of the net, and hence common to two circuits.

Positive Direction of a Section: The direction in which the correction is to be computed (regardless of whether the difference in elevation is positive or negative).

Direction Arrows.—"Main direction arrows" are those used to designate the positive directions of the sections. In general, these direction arrows serve also to indicate the direction in which the closing error of a circuit is calculated. In certain interior sections, however, the latter direction may be contrary to the former, and is then distinguished by means of an "auxiliary direction arrow," which is written beside the section and within the circuit in question, as shown in Fig. 1.

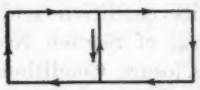

FIG. 1.

Weights.—The weight, p, of an observed difference in elevation in a given section depends upon the length of the section, the number of set-ups it contains, the number of independent measurements entering into the data, and the quality of instruments and survey methods used. On ordinary slopes, the number of set-ups is approximately proportional to the length of the line, and, therefore, it is customary to weight the observations in inverse proportion to the distance (which may be determined by stadia measurements, by pacing, or by scaling from a map). Moreover, the weight is directly proportional to the number of times the levels were run; that is, a difference in elevation which is determined from the mean of two separate runnings of a section is given twice as much weight as a difference in elevation obtained from a single running.

Effective Lengths.—It is convenient, in the problem of the level net, to deal with weights in terms of the "effective lengths" of sections. The effective length of a section is defined as the reciprocal of the weight; that is, $\frac{1}{p}$; consequently, it is directly proportional to the length of the section and inversely proportional to the number of measurements.

Closure Conditions.—Two types of closure will be considered, namely, the circuit closure and the route closure. By a "circuit-closure condition"

is meant the requirement that the sum of the section corrections in a given circuit must be equal, and opposite in sign, to the closing error of that circuit.

If the level net contains two or more control bench-marks, conditions of route closure arise. By a "route-closure condition" is meant the requirement that the section corrections along an arbitrary route connecting two given control bench-marks must have a predetermined sum, equal (and opposite in sign) to the closing error along that route; this situation gives rise to the following term:

Conditioned Route: An arbitrary route, selected for purposes of adjustment, connecting two control bench-marks.

Condition Multipliers.—"Condition multipliers" are undetermined coefficients which are used to represent the closure conditions to which a section is subject. For example, Closure Conditions Nos. I, II, etc., are represented by the condition multipliers, C_{I}, C_{II}, etc., respectively. By this means, the correction for any section may be expressed as a linear function of the closure conditions.

Section Corrections.—The corrections to be determined for the observed differences in elevation along the various sections are called "section corrections." They are denoted by v_1, v_2, etc., the subscripts of which refer to Sections Nos. 1, 2, etc., respectively. By means of the theory of least squares it is found (see Appendix I) that the section correction for a given section is equal to the product of the effective length of the section by the sum of its condition multipliers. Expressed algebraically, the section correction, v_m, of Section No. m, with an effective length of $L'm$, which is subject to Closure Conditions Nos. I, II, etc., is given by the equation,

$$v_m = L'm\ (C_{\text{I}} + C_{\text{II}} + \ldots) \qquad (1)$$

The number of closure conditions governing a given section is equal to the number of circuits or conditioned routes to which the section belongs.

Normal Equations.—Equations of the type obtained when the section corrections as expressed in Equation (1) are summed up around a circuit or along a conditioned route and placed equal, and opposite in sign, to the closing error, are known as "normal equations." The number of normal equations in any net is equal to the number of circuits, plus the number of conditioned routes.

Description of Method 1; Example No. 1

For purposes of description, the solution will be divided into the following steps: (*a*) Preparation of plan for the adjustment, and calculation of closing errors; (*b*) formation and solution of normal equations; (*c*) calculation of section corrections and verification of circuit closures; and, (*d*) calculation of adjusted elevations of bench-marks.

To illustrate the detailed procedure, two examples will be given, the first involving only circuit-closure conditions, and the second entailing a route-closure condition.

The net chosen to exemplify Example No. 1 is illustrated in Fig. 2. It contains four circuits and nine bench-marks, five of which are junction points, and the remaining four, intermediate bench-marks. Point *A* will be assumed to be a control bench-mark, with an elevation of 416.15 ft. The approximate distances between bench-marks are shown in Fig. 2.

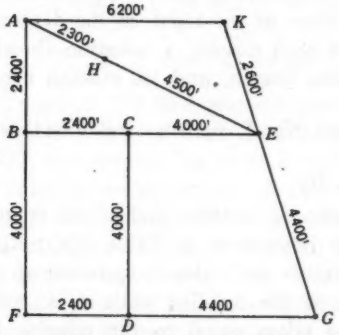

FIG. 2.—SKETCH OF LEVEL NET.

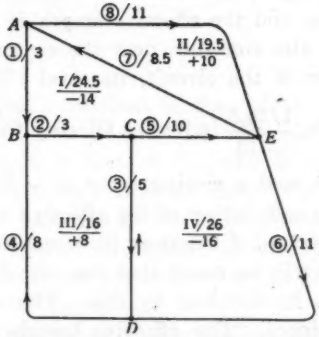

FIG. 3.—PLAN FOR ADJUSTMENT.

The observational data are given in Table 1, Appendix II, and consist of field notes covering levels which were run over six routes, as follows: Route No. 1: *A* to *B* to *C* and return; Route No. 2: *C* to *D* to *G* to *E* and return; Route No. 3: *D* to *F* to *B* and return; Route No. 4: *C* to *E* (one way); Route No. 5: *A* to *H* to *E* and return; and, Route No. 6: *A* to *K* to *E* and return.

(a) Preparation of Plan for the Adjustment, and Calculation of Closing Errors.—The preliminary details necessary to put the data in readiness for performing the adjustment include the preparation of a plan and the determination of the closing errors, and may be performed in the following order:

(1) Prepare a plan showing the junction points and control bench-marks, and the connecting routes over which the leveling was executed;

(2) Assign a system of direction arrows;

(3) Number the sections, determine their effective lengths, and record these values on the plan;

(4) From the field data, calculate the observed difference in elevation over each section in the positive direction (that is, as shown by the main direction arrow); also, determine the observed differences in elevation over the sub-sections, which will be required in computing the adjusted elevations of the intermediate bench-marks; and,

(5) Number the circuits, calculate their total effective lengths and their closing errors, and record these values on the plan.

The plan for the adjustment for Example No. 1 is shown in Fig. 3, intermediate bench-marks being omitted. Direction arrows are assigned in such a way as to secure continuous routing through all the circuits with a

minimum number of auxiliary direction arrows. This system of marking the sections possesses the following advantages: (1) It fixes the route to be taken in calculating the closing error of each circuit; (2) it determines the direction in which each section correction is to be calculated; and (3), it enables the normal equations to be formed by inspection.

In Fig. 3, the number of the section is shown by the figure enclosed within a circle, and the effective length is written to the right of the diagonal bar. Fig. 3 also contains, near the center of each circuit, a notation showing the number of the circuit, its total effective length, and its closing error (for example, $\frac{\text{I}/24.5}{-14}$ indicates Closing Circuit No. I, with a total effective length of 24.5, and a closing error of -0.14 ft).

The calculation of the effective lengths of sections and of the corresponding observed differences in elevation is contained in Table 2(*a*), Appendix II. It will be noted that the actual lengths are reduced to terms of relative lengths by dividing by 800. The use of the smaller units is a matter of convenience. The effective lengths are taken equal to the relative lengths of the corresponding sections, except for Section No. 5. Since this section was leveled only once (all other sections having been leveled twice), its effective length is twice its relative length.

In explanation of the calculation of the observed differences in elevation, consider Section No. 1, which extends from Point *A* to Point *B* in Fig. 2. The field notes, given in Table 1, Appendix II, show that this section is a part of Route No. 1 and was run both forward and back. In the forward running the observed difference in elevation was $422.59 - 416.15 = +6.44$; in the back running the result was $416.17 - 422.65 = -6.48$. In this instance the positive direction of the section (as shown by the direction arrow in Fig. 3) coincides with the forward running, and, consequently, the observed (mean) difference in elevation is + 6.46 ft (Column (8), Table 2(*a*), Appendix II). (In taking the arithmetic mean of an even number and an odd number, the even value is chosen for convenience; for example, in Section No. 7, Table 2(*a*), the mean of 9.20 and 9.07 is taken as 9.14). Table 2(*b*) contains the calculation of the observed differences in elevation over the sub-sections.

The calculation of the closing errors is contained in Table 3, Appendix II. The closing error of a given circuit is simply the sum of the observed differences in elevation of its component sections, taken in a continuous direction around the circuit in accordance with the direction arrows. The closure conditions corresponding to the various circuits are numbered in the same way as the circuits themselves, and, hence, the condition multipliers for Circuits Nos. I, II, III, etc. (Fig. 3), are C_{I}, C_{II}, C_{III}, etc., respectively.

(*b*) *Formation of Normal Equations.*—In accordance with Equation (1), the section corrections, taken in a continuous direction around the various circuits, are as shown in Table 4, Appendix II. When the sums of these corrections are placed equal, and opposite in sign, to the closing errors of

the respective circuits (0.01 ft being taken, for convenience as a unit), the following normal equations are obtained:

$$24.5\ C_{\text{I}} + 8.5\ C_{\text{II}} + 3\ C_{\text{III}} + 10\ C_{\text{IV}} = +\ 14 \quad \text{(I)}$$

$$8.5\ C_{\text{I}} + 19.5\ C_{\text{II}} = -\ 10 \quad \text{(II)}$$

$$3\ C_{\text{I}} + 16\ C_{\text{III}} - 5\ C_{\text{IV}} = -\ 8 \quad \text{(III)}$$

and,

$$10\ C_{\text{I}} - 5\ C_{\text{III}} + 26\ C_{\text{IV}} = +\ 16 \quad \text{(IV)}$$

In Equation (I) the greatest numerical coefficient is that of C_{I}, and is equal to the total effective length of Circuit No. I; likewise, in Equation (II) the greatest numerical coefficient is that of C_{II}, and it is equal to the total effective length of Circuit No. II; and, similarly, for Equations (III) and (IV). These quantities (namely, 24.5 C_{I} in Equation (I), 19.5 C_{II} in Equation (II), etc.) will be referred to as the "major" terms of their respective equations. The remainder of the quantities on the left-hand side of the equations will be called "minor" terms.

As a matter of fact, it is simpler to write the normal equations at once as in Table 5(*a*), Appendix II. They are written in columns which are numbered to correspond to the various closure conditions, the individual terms being entered in the lines with numbers that agree with the subscripts of the condition multipliers. The major terms (which are underscored) thus appear in diagonal formation. As already stated, their coefficients are merely the total effective lengths of the respective circuits (or conditioned routes).

Minor terms arise from sections which are subject to two (or more) closure conditions, and their coefficients are equal to the effective lengths of the sections involved. For example, in writing the minor terms of Equation (I), it is seen from Fig. 3 that Circuit No. I is tied to Circuit No. II by Section No. 7, with effective length, 8.5; to Circuit No. III by Section No. 2, with effective length, 3; and to Circuit No. IV by Section No. 5, with effective length, 10. Consequently, in Column (I), Table 5, Appendix II, 8.5 is entered in Line II, 3 in Line III, and 10 in Line IV.

In the case of sections marked with an auxiliary direction arrow the minor term is always prefixed with the negative sign. The constant, or right-hand, term of the equation is entered at the bottom of the appropriate column.

Checking the Normal Equations.—As the coefficients of the minor terms always appear in two of the normal equations, it is desirable to utilize this fact as a check. The coefficients should read the same vertically and horizontally.

Solution of Normal Equations.—From the following typical system of equations, in three unknowns, namely,

$$AC_{\text{I}} + bC_{\text{II}} + cC_{\text{III}} = k_1 \quad \text{(Ia)}$$

$$bC_{\text{I}} + BC_{\text{II}} + dC_{\text{III}} = k_2 \quad \text{(IIa)}$$

and,

$$cC_{\text{I}} + dC_{\text{II}} + CC_{\text{III}} = k_3 \ldots\ldots\ldots\ldots\ldots\ldots (\text{III}a)$$

Equations (Ib), (IIb), and (IIIb) may be written, thus:

$$AC_{\text{I}} = k_1 - (bC_{\text{II}} + cC_{\text{III}}) \ldots\ldots\ldots\ldots\ldots\ldots (\text{I}b)$$

$$BC_{\text{II}} = k_2 - (dC_{\text{III}} + bC_{\text{I}}) \ldots\ldots\ldots\ldots\ldots\ldots (\text{II}b)$$

and,

$$CC_{\text{III}} = k_3 - (cC_{\text{I}} + dC_{\text{II}}) \ldots\ldots\ldots\ldots\ldots\ldots (\text{III}b)$$

respectively. Let, $C_{\text{I}} = C_{\text{I-1}} + C_{\text{I-2}} + C_{\text{I-3}} + \text{etc.}$; $C_{\text{II}} = C_{\text{II-1}} + C_{\text{II-2}} + C_{\text{II-3}} + \text{etc.}$; and, $C_{\text{III}} = C_{\text{III-1}} + C_{\text{III-2}} + C_{\text{III-3}} + \text{etc.}$

Upon substituting these values of C_{I}, C_{II}, and C_{III} into Equations (Ib), (IIb), and (IIIb), the following expressions, respectively, are obtained:

$$AC_{\text{I-1}} + AC_{\text{I-2}} + AC_{\text{I-3}} + \ldots = k_1 - (bC_{\text{II-1}} + cC_{\text{III-1}}) - (bC_{\text{II-2}} + cC_{\text{III-2}}) - \ldots\ldots\ldots\ldots\ldots\ldots (\text{I}c)$$

$$BC_{\text{II-1}} + BC_{\text{II-2}} + BC_{\text{II-3}} + \ldots = (k_2 - bC_{\text{I-1}}) - (dC_{\text{III-1}} + bC_{\text{I-2}}) - (dC_{\text{III-2}} + bC_{\text{I-3}}) - \ldots\ldots\ldots\ldots\ldots\ldots (\text{II}c)$$

and,

$$CC_{\text{III-1}} + CC_{\text{III-2}} + CC_{\text{III-3}} + \ldots = (k_3 - cC_{\text{I-1}} - dC_{\text{II-1}}) - (cC_{\text{I-2}} + dC_{\text{II-2}}) - (cC_{\text{I-3}} + dC_{\text{II-3}}) - \ldots\ldots\ldots\ldots (\text{III}c)$$

A solution by parts (if convergent) may now be made, one term at a time being computed from each of the foregoing equations. Its value is then used in the succeeding equations to find the next term, in the following manner: $AC_{\text{I-1}} = k_1$; $BC_{\text{II-1}} = k_2 - bC_{\text{I-1}}$; $CC_{\text{III-1}} = k_3 - (cC_{\text{I-1}} + dC_{\text{II-1}})$; $AC_{\text{I-2}} = -(bC_{\text{II-1}} + cC_{\text{III-1}})$; $BC_{\text{II-2}} = -(dC_{\text{III-1}} + bC_{\text{I-2}})$; $CC_{\text{III-2}} = -(cC_{\text{I-2}} + dC_{\text{II-2}})$; $AC_{\text{I-3}} = -(bC_{\text{II-2}} + cC_{\text{III-2}})$; $BC_{\text{II-3}} = -(dC_{\text{III-2}} + bC_{\text{I-3}})$; $CC_{\text{III-3}} = -(cC_{\text{I-3}} + dC_{\text{II-3}})$; etc. Quantities of this type will be referred to as "principal terms;" they comprise both "preliminary values" (or terms found in the first set of computations, as, for example, $AC_{\text{I-1}}$, $BC_{\text{II-1}}$, $CC_{\text{III-1}}$) and "increments" (or terms similar to $AC_{\text{I-2}}$, $BC_{\text{II-2}}$, $CC_{\text{III-2}}$, etc.), which result from succeeding sets of computations.

This method appears to be applicable to the vast majority of normal equations encountered in practical engineering problems, and possesses the important advantage that, in dealing with equations containing four or more unknowns, the solution is usually obtained in appreciably less time than that required by a precise method. Although, generally, a small degree of approximation is involved, the results are practically exact in many cases.

The number of sets of computations required to obtain an accurate solution is usually from four to six; occasional examples may require one or two additional sets. If the convergence is rapid, the computations may be continued until the increments become negligible. The principal terms in

each equation are then totaled, and the value of the unknown is found by dividing the resulting sum by the coefficient of the major term of the equation.

If the convergence is relatively slow, it generally happens that, after a few sets of computations are completed, the increments in the separate equations begin to converge with ratios that tend to be nearly equal. When this stage is reached, the solution can be expedited greatly, without appreciable sacrifice of precision, by summing up the remaining increments as geometrical series. This expedient may also be utilized in connection with the more rapidly converging equations, provided the ratios of convergence in the separate equations are approximately equal.

Before effecting the summation, in any case, it is desirable that the ratios of convergence in the separate equations be brought to agreement within fairly definite limits, depending upon the desired degree of precision. The writer's experience indicates that if the range of variation in these ratios does not materially exceed 0.05 (as will presently be illustrated by numerical examples), the remaining increments can be summed up as geometrical series with results sufficiently accurate to reduce the closing errors of the individual circuits to considerably less than 0.01 ft.

The process of summing up the remaining terms of the various series of increments consists of multiplying the last set of calculated increments by the summation factor. These summations are then added to the principal terms previously calculated, and, finally, the unknowns are found by dividing the resulting totals by the coefficients of the major terms of the respective equations. The summation factor of a geometrical series is equal to $\frac{r}{1-r}$, in which r is the ratio of convergence of successive terms. For present requirements, the summation factor will be computed by means of an "average ratio of convergence."

If the increments in the separate equations attain the desired degree of uniformity of convergence in five or less sets of computations, the average ratio of convergence is given, with satisfactory precision, by:

$$r = \frac{s_n}{s_{n-1}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which s_n is the absolute sum of the increments in the final set of computations, and s_{n-1} is the absolute sum of the increments in the preceding set.

If more than five sets of computations are required before the ratios attain the desired degree of uniformity, a better approximation of the average ratio of convergence is generally obtained by using the last three sets of results, as follows:

$$r' = \frac{r^2}{r_{n-1}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

in which r is as defined in Equation (2), and $r_{n-1} = \frac{s_{n-1}}{s_{n-2}}$. The corresponding summation factor is then taken as $\frac{r'}{1-r'}$.

Referring to Table 5(*a*), Appendix II, the calculations incident to the solution of the equations of Example No. 1 are, as follows: In Line 1, the constant term of Equation (I) (namely, 14.00), is taken as the preliminary value of 24.5 C_{I}. It is entered as a "principal term," is underscored, and is extended horizontally in proportion to the coefficients in Line 1, Table 5(*a*), as follows:

$$\frac{8.5}{24.5}(14.00) = 4.85 \text{ (Column (II))}$$

$$\frac{3}{24.5}(14.00) = 1.71 \text{ (Column (III))}$$

and,

$$\frac{10}{24.5}(14.00) = 5.71 \text{ (Column (IV))}$$

For the preliminary value of 19.5 C_{II} in Line 2 the value found for Line 1, Column (II), is subtracted from the constant term of Equation (II), giving $-10.00 - 4.85 = -14.85$. This quantity likewise constitutes a principal term, and is underscored and extended horizontally in proportion to the coefficients in Line II, giving:

$$\frac{8.5}{19.5}(-14.85) = -6.48 \text{ (Column (I))}$$

The preliminary value of 16 C_{III} in Line 3 is determined by subtracting from the constant term of Equation (III) the value calculated for Line 1, Column (III) of Table 5(*a*), which gives the following result: $-8.00 - 1.71 = -9.71$. This quantity, in turn, is extended horizontally in proportion to the coefficients in Line III.

In a similar manner, the preliminary value of 26 C_{IV} in Line 4, Table 5(*a*), is calculated by subtracting the values in Line 1, Column (IV), and Line 3, Column (IV), from the constant term of Equation (IV), giving: $16.00 - (5.71 + 3.04) = 7.25$, which is then extended in the same fashion. This completes the first set of computations.

In succeeding sets (Lines 5 to 16), the principal terms are obtained by taking the negative sum of all the values which occur below the principal term in the preceding set; for example, in Line 5, the principal term, or increment of 24.5 C_{I}, is: $-(-6.48 - 1.82 + 2.79) = 5.51$.

At the end of the fourth set of computations, the ratios of convergence in the separate columns of Table 5(*a*) (obtained by dividing the increments of the fourth set by those of the third set) are as follows: $\frac{\text{I}}{0.429}$, $\frac{\text{II}}{0.431}$, $\frac{\text{III}}{0.404}$, and $\frac{\text{IV}}{0.423}$. From the foregoing quantities, the maximum range of variation is found to be $0.431 - 0.404 = 0.027$. Although these ratios are quite uniform, it is obvious that several additional sets of computations would be required in order to reduce the increments to negligible quantities; the remaining increments, therefore, are summed up as geometrical series.

The average ratio of convergence, as given by Equation (2), is $\frac{1.41}{3.33}$ = 0.423, and the corresponding summation factor is 0.734. The resulting summations (obtained by multiplying the fourth set of increments by the summation factor) appear in Line 17, Table 5(*a*). Line 18 contains the sums of the principal terms (including the quantities in Line 17), and the values of the unknowns (the condition multipliers), are given in Line 19.

The calculations of Table 5(*a*) were performed with the aid of a 10-in. slide-rule. The results of this solution, when compared with values obtained from a precise solution,[3] are all found to be correct to three decimal places, as shown.

(c) Calculation of Section Corrections and Verification of Circuit Closures.—Table 6(*a*), Appendix II, contains the computation of the section corrections, which is developed in accordance with Equation (1). This step is facilitated by tabulating the condition multipliers in groups, according to the sections composing the successive circuits. For example (see Fig. 3), C_{I} is applied to Sections Nos. 1, 2, 5, and 7; C_{II}, to Sections Nos. 7 and 8; C_{III}, to Sections Nos. 2, 3, and 4; and C_{IV}, to Sections Nos. 3 (with sign reversed), 5, and 6. The sum of the condition multipliers for each section appears in Column (5), Table 6(*a*), and the section corrections (the products of Column (2) and Column (5) to the nearest tenth of a unit, corresponding to the thousandth of a foot), are given next. In Column (7) the adopted section corrections are recorded to the nearest 0.01 ft.

The verification of the circuit closures is given in Table 7(*a*), Appendix II, which indicates that the section corrections as calculated will reduce the closing errors to zero.

(d) Calculation of Adjusted Elevations of Bench-Marks.—The calculation of the adjusted elevations of the bench-marks is given in Table 8(*a*), Appendix II. In dealing with intermediate bench-marks, the adjusted difference in elevation over each sub-section is calculated by prorating the section correction in the ratio of the length of the sub-section to that of the section (see Table 2(*b*), Appendix II).

Description of Method 1; Example No. 2

The data contained in Table 1, Appendix II, will also be used in connection with Example No. 2, with the added requirements that both Points *A* and *C* shall be treated as control bench-marks, with elevations of 416.15 and 424.01, respectively. For purposes of adjustment, it is theoretically immaterial which path connecting these points is taken as the conditioned route. The selected route is that consisting of Sections Nos. 1 and 2.

The closing errors of the four circuits are as given in Table 3, Appendix II; that of the conditioned route is found by subtracting the required difference in elevation from the observed difference in elevation, and is equal to 7.90 − 7.86 = + 0.04.

[3] For a general method of obtaining a precise solution of normal equations, see "Application of the Theory of Least Squares to the Adjustment of Triangulation," by Oscar S. Adams, *Special Publication No. 28*, U. S. Coast and Geodetic Survey.

The new system of normal equations, together with their solution, is given in Table 5(*b*), Appendix II. Equation (V) expresses the closure requirements of the conditioned route. It will be noted that the inclusion of the added closure condition requires that the term, 6 C_V, be included in Equation (I), and the term, 3 C_V, in Equation (III).

At the end of the fifth set of computations, the increments in each column have begun to converge, with the exception of Column (III). By carrying the calculations through one additional line, it will be found that the next increment in that column is -0.06, from which it may be concluded that the first five sets of computations are sufficient, provided the ratios of convergence are sufficiently uniform. These ratios are, as follows:

$$\frac{\text{I}}{0.557}, \frac{\text{II}}{0.554}, \frac{\text{III}}{0.600}\left(\text{using } \frac{0.06}{0.10}\right), \frac{\text{IV}}{0.564}, \text{ and } \frac{\text{V}}{0.540},$$

from which it is seen that the maximum difference is approximately 0.06.

The final results given in Table 5(*b*), when compared with a precise solution, are found to be correct to three decimal places, except for C_V, which is in error by 0.003.

Table 6(*b*), Appendix II, contains the calculation of the section corrections, and Table 7(*b*), Appendix II, the verification of the closures. The calculation of the adjusted elevations of the bench-marks is given in Table 8(*b*), Appendix II.

Description of Method 2

The second method of adjusting a level net differs from the first in the following respects: (1) Instead of the writing and solution of the normal equations, the errors in the various circuits are successively distributed, carried across, and re-distributed; (2) the section corrections are found by direct addition of the distributed quantities; and (3) all these calculations are entered directly on the plan for the adjustment.

In order to illustrate this method, the closing errors of the previous examples will be adjusted. The explanations will be confined to the determination of the section corrections, as the preliminary and final details are the same in both solutions. Moreover, the results are theoretically identical in the two methods.

Example No. 1.—The calculation of the section corrections is given in Fig. 4, and the errors requiring distribution in the different cycles are listed in Table 9(*a*), Appendix II.

Table 10, Appendix II, which has been especially prepared for the purpose of assisting in the explanation of the method, contains complete details of all the calculations involved in the first two cycles of the adjustment. The closing error of each circuit is distributed (with change in sign) among the several sections of the circuit in proportion to their effective lengths. Corrections thus applied to sections subject to two or more closure conditions constitute new errors in the adjoining circuits (or conditioned routes), and, therefore, are carried across at the first opportunity; in the

first cycle of adjustments, they are added to the original closing error of the adjoining circuit, the combined closing error being then distributed.

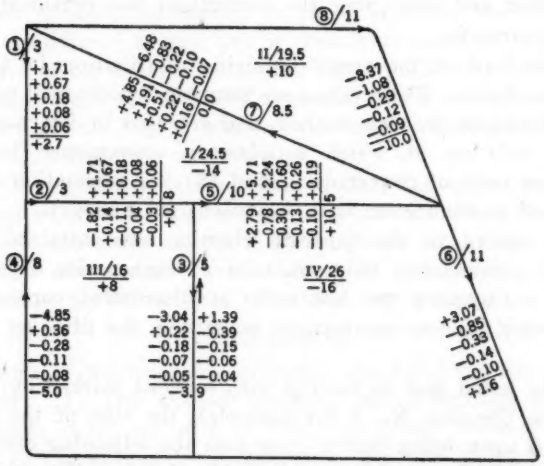

FIG. 4.—ADJUSTMENT OF LEVEL NET BY SUCCESSIVE DISTRIBUTION OF THE CLOSING ERRORS, EXAMPLE NO. 1.

Referring to Table 10(*a*), the closing error of Circuit No. I (namely, −14.00), is distributed as follows: Section No. 1; $\frac{3}{2.45}(+14.00) = +1.71$; Section No. 2: $\frac{3}{24.5}(+14.00) = +1.71$; Section No. 5: $\frac{10}{24.5}(+14.00) = +5.71$; and, Section No. 7: $\frac{8.5}{24.5}(+14.00) = +4.85$.

In Circuit No. II, the foregoing correction to Section No. 7, is carried across and added to the original closing error, giving: $+4.85 + 10.00 = +14.85$; this quantity is then distributed as follows: Section No. 7: $\frac{8.5}{19.5}(-14.85) = -6.48$; and, Section No. 8: $\frac{11}{19.5}(-14.85) = -8.37$.

In Circuit No. III, the correction already applied to Section No. 2 is taken up, and the combined error (namely, $+1.71 + 8.00 = +9.71$) is distributed in the following proportions: Section No. 2: $\frac{3}{16}(-9.71) = -1.82$; Section No. 3: $\frac{5}{16}(-9.71) = -3.04$; and, Section No. 4: $\frac{8}{16}(-9.71) = -4.85$.

In Circuit No. IV, the corrections previously assigned to Sections Nos. 3 and 5 are carried across, and the error requiring distribution is, therefore, $+3.04 + 5.71 - 16.00 = -7.25$, which, in turn, is distributed (with change in sign) in proportion to the effective lengths of the sections which compose this circuit.

The circuits are adjusted in a fixed sequence, and the same process of distribution is continued in succeeding cycles, the closing errors being found by bringing over and combining the corrections not previously transferred from adjacent circuits.

A record is kept of the errors requiring distribution in the successive steps (see Table 9(*a*)). These values are numerically identical with the "principal terms" found in the first method, and are used to determine: (1) The stage in the solution at which satisfactory convergence is established; (2) the average ratio of convergence; and (3) the summation factor, previously mentioned in connection with Equations (2) and (3).

When the errors in the separate circuits have attained the desired uniformity of convergence, the remainder of each series of corrections is computed by multiplying the last cycle of distributed corrections by the summation factor. These summations constitute the fifth set of values in Fig. 4.

It should be noted that in dealing with sections marked by the auxiliary direction arrow (Section No. 3 for example), the sign of the correction is always changed upon being carried over into the adjoining circuit; furthermore, the signs of all quantities on that side of the section which bears the auxiliary direction arrow are reversed when the individual corrections of a section are totaled.

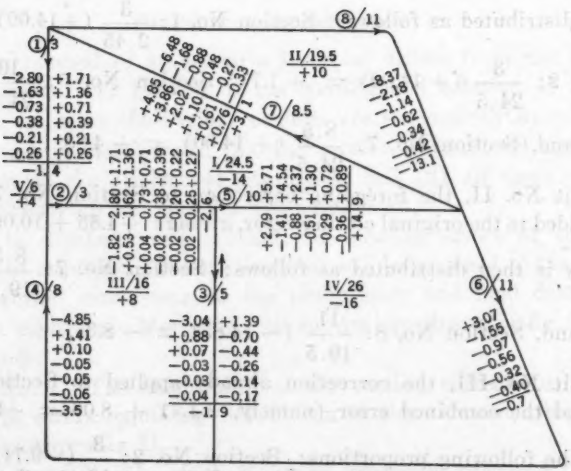

FIG. 5.—ADJUSTMENT OF LEVEL NET BY SUCCESSIVE DISTRIBUTIONS OF THE CLOSING ERRORS, EXAMPLE NO. 2.

Example No. 2.—The calculation of the section corrections for Example No. 2 is given in Fig. 5. The errors requiring distribution in the successive steps are shown in Table 9(*b*), Appendix II. Table 10(*b*), Appendix II, contains an explanation of the manner of determining the errors to be distributed in the first two cycles, together with their distributions (the first cycle in this case being the same for Examples Nos. 1 and 2.)

In adjusting the conditioned route, the closing error is found by bringing across and combining the errors arising from the preceding distributions in the contiguous circuits (including, in the first cycle, the original closing error of the conditioned route), and is then distributed in proportion to the effective lengths of the sections composing the conditioned route. For exterior sections (Section No. 1 for example), such corrections are entered on the outer side of the section (see Fig. 5) whereas, for interior sections (Section No. 2), they are written astride the section.

In turn, these corrections are taken up into the adjacent circuits when the closing errors of the latter are next calculated. Since interior sections of a conditioned route are subject to three closure conditions, all corrections applied to such sections are carried across in two places. In Fig. 5, the sixth set of values are those found by summation of series.

Conclusions

As the result of the foregoing examples, the following conclusions appear to be justified: (1) The methods presented are sufficiently precise to meet the requirements of the majority of level nets, except those of large geodetic surveys; and (2), in view of the comparative ease with which a correct adjustment can be effected, and of the obvious advantages of thus eliminating the uncertainty and confusion which otherwise prevail, no carefully executed system of bench-marks should be considered complete until the elevations have been properly adjusted.

APPENDIX I

Theory

According to the theory of least squares, the sum, Σpv,[2] is to be made a minimum, subject to the given closure conditions. Using the data of Example No. 1 (Tables 1 to 3, inclusive, Appendix II), the function which is to be made a minimum may be expressed as follows:[4]

$$F = \frac{v_1^2}{3} + \frac{v_2^2}{3} + \frac{v_3^2}{5} + \frac{v_4^2}{8} + \frac{v_5^2}{10} + \frac{v_6^2}{11} + \frac{v_7^2}{8.5} + \frac{v_8^2}{11}$$
$$- 2\,C_{\mathrm{I}}\,(v_1 + v_2 + v_5 + v_7 - 14) - 2\,C_{\mathrm{II}}\,(v_7 + v_8 + 10)$$
$$- 2\,C_{\mathrm{III}}\,(v_2 + v_3 + v_4 + 8) - 2\,C_{\mathrm{IV}}\,(- v_3 + v_5 + v_6 - 16) \ldots . (4)$$

When the partial derivatives of this function with respect to v_1, v_2, etc., are equated to zero, the following expressions for the section corrections are obtained: $v_1 = 3\,C_{\mathrm{I}}$; $v_2 = 3\,(C_{\mathrm{I}} + C_{\mathrm{III}})$; $v_3 = 5\,(C_{\mathrm{III}} - C_{\mathrm{IV}})$; $v_4 = 8\,C_{\mathrm{III}}$; $v_5 = 10\,(C_{\mathrm{I}} + C_{\mathrm{IV}})$; $v_6 = 11\,C_{\mathrm{IV}}$; $v_7 = 8.5\,(C_{\mathrm{I}} + C_{\mathrm{II}})$; and, $v_8 = 11\,C_{\mathrm{II}}$.

The foregoing results may be summarized in the statement that the section correction for a given section is equal to the product of the effective length of the section by the sum of its condition multipliers.

[4] "Some Elementary Examples of Least Squares," by Oscar S. Adams, *Serial No. 250*, U. S. Coast and Geodetic Survey.

APPENDIX II

TABLES

The tables in Appendix II are each introduced and explained adequately in the text of the paper.

TABLE 1.—DATA FROM FIELD NOTES

Route No. 1		Route No. 2		Route No. 3		Route No. 4		Route No. 5		Route No. 6	
Point	Elevation	Point	Elevation	Point	Elevation	Point	Elevation	Point	Elevation	Point	Elevation
A	416.15	*C*	424.06	*D*	421.66	*C*	424.06	*A*	416.15	*A*	416.15
B	422.59	*D*	421.70	*F*	424.11	*E*	425.16	*H*	418.22	*K*	410.02
C	424.06	*G*	428.65	*B*	422.59	....	.....	*E*	425.35	*E*	425.34
B	422.65	*E*	425.25	*F*	424.05	....	.....	*H*	418.32	*K*	409.94
A	416.17	*G*	428.58	*D*	421.58	....	.....	*A*	416.28	*A*	416.04
....	.....	*D*	421.62	....	.....	....	.....	....	.....	....	.....
....	.....	*C*	423.92	....	.....	....	.....	....	.....	....	.....

TABLE 2.—COMPUTATION OF EFFECTIVE LENGTHS AND CORRESPONDING OBSERVED DIFFERENCES IN ELEVATION

	(*a*) Relating to Sections							(*b*) Relating to Sub-Sections				
Section	Between Points:	Distance, *l*, in feet	Relative length	Effective length, *L'*	Field Result s*		Observed difference in elevation †	Sub-section: between points:	Ratio, length of sub-section to length of section	Field Results *		Observed difference in elevation †
					Forward	Back				Forward	Back	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
1	*A* and *B*.....	2 400	3	3	+6.44	−6.48	+6.46	.....	.....	.....	.....	.....
2	*B* and *C*.....	2 400	3	3	+1.47	−1.41	+1.44	.....	.....	.....	.....	.....
3	*C* and *D*.....	4 000	5	5	−2.36	+2.30	−2.33	.....	.....	.....	.....	.....
4	*D* and *B* (*via F*)..	6 400	8	8	+0.93	−1.01	+0.97	*D* and *F*...	24 :64	+2.45	−2.47	+2.46
5	*C* and *E*.....	4 000	5	10	+1.10	.....	+1.10	.....	.....	.....	.....	.....
6	*E* and *D* (*via G*).	8 800	11	11	+3.55	−3.63	−3.59	*E* and *G*...	44 :88	−3.40	+3.33	+3.36
7	*E* and *A* (*via H*).	6 800	8.5	8.5	+9.20	−9.07	−9.14	*E* and *H*..	45 :68	+7.13	−7.03	−7.08
8	*A* and *E* (*via K*).	8 800	11	11	+9.19	−9.30	+9.24	*A* and *K*..	62 :88	−6.13	+6.10	−6.12

* In the direction actually run. † In the positive direction.

TABLE 3.—COMPUTATION OF CLOSING ERRORS (OBSERVED DIFFERENCES IN ELEVATION)

Section No.	Circuit No. I	Circuit No. II	Circuit No. III	Circuit No. IV
1.....	+6.46	.....	.....	.....
2.....	+1.44	.....	+1.44	.....
3.....	.....	.....	−2.33	+2.33*
4.....	.....	.....	+0.97	.....
5.....	+1.10	.....	.....	+1.10
6.....	.....	.....	.....	−3.59
7.....	−9.14	−9.14	.....	.....
8.....	.....	+9.24	.....	.....
	+9.00	+9.24	+2.41	+3.43
	−9.14	−9.14	−2.33	−3.59
Summation.....	−0.14	+0.10	+0.08	−0.16

* This value taken in the negative direction of Section No. 3; that is, as shown by the auxiliary direction arrow in Fig. 3.

TABLE 4.—Formation of Normal Equations

Circuit No. I	Circuit No. II	Circuit No. III	Circuit No. IV
$v_1 = 3C_I$	$v_7 = 8.5\ (C_{II} + C_I)$	$v_2 = 3\ (C_{III} + C_I)$	$-v_3 = 5\ (C_{IV} - C_{III})$
$v_2 = 3\ (C_I + C_{III})$	$v_8 = 11\ C_{II}$	$v_3 = 5\ (C_{III} - C_{IV})$	$v_8 = 10\ (C_{IV} + C_I)$
$v_5 = 10\ (C_I + C_{IV})$	.....	$v_4 = 8\ C_{III}$	$v_6 = 11\ C_{IV}$
$v_7 = 8.5\ (C_I + C_{II})$	.....	.....	.....

TABLE 5.—Solution of Normal Equations

(*a*) Example No. 1

Line	I	II	III	IV
I	24.5	8.5	3	10
II	8.5	19.5	.....	.....
III	3	.....	16	−5
IV	10	.....	−5.	26
...	.....	.....	.....	.....
k	14	−10	−8	16
1	14.00	4.85	1.71	5.71
2	−6.48	−14.85	.....	.....
3	−1.82	.....	−9.71	3.04
4	2.79	.....	−1.39	7.25
5	5.51	1.91	0.67	2.24
6	−0.83	−1.91	.....	.....
7	0.14	.....	0.72	−0.22
8	−0.78	.....	0.39	−2.02
9	1.47	0.51	0.18	0.60
10	−0.22	−0.51	.....	.....
11	−0.11	.....	−0.57	0.18
12	−0.30	.....	0.15	−0.78
13	0.63	0.22	0.08	0.26
14	−0.10	−0.22	.....	.....
15	−0.04	.....	−0.23	0.07
16	−0.13	.....	0.06	−0.33
17	0.46	−0.16	−0.17	−0.24
18	22.07	−17.65	−9.96	3.88
19	0.901	−0.905	−0.622	0.149

Example No. 1:

$s_4 = 1.41$; $s_3 = 3.33$; $r = 0.423$; and, $\frac{r}{1-r} = 0.734$

Example No. 2:

$s_3 = 3.64$; $s_4 = 6.47$; $r = 0.562$; and, $\frac{r}{1-r} = 1.28$

(*b*) Example No. 2

I	II	III	IV	V
24.5	8.5	3	10	6
8.5	19.5	.....	.....	.....
3	.....	16.	−5	3
10	.....	−5	26	.....
6	.....	3	.....	6
14	−10	−8	16	−4
14.00	4.85	1.71	5.71	3.42
−6.48	−14.85	.....	.....	.....
−1.82	.....	−9.71	3.04	−1.82
2.79	.....	−1.39	7.25	.....
−5.60	.....	−2.80	.....	−5.60
11.11	3.86	1.36	4.54	2.72
−1.68	−3.86	.....	.....	.....
0.53	.....	2.83	−0.88	0.53
−1.41	.....	0.70	−3.66	.....
−3.25	.....	−1.62	.....	−3.25
5.81	2.02	0.71	2.37	1.42
−0.88	−2.02	.....	.....	.....
0.04	.....	0.21	−0.07	0.04
−0.88	.....	0.44	−2.30	.....
−1.46	.....	−0.73	.....	−1.46
3.18	1.10	0.39	1.30	0.78
−0.48	−1.10	.....	.....	.....
−0.02	.....	−0.10	0.03	−0.02
−0.51	.....	0.26	−1.33	.....
−0.76	.....	−0.38	.....	−0.76
1.77	0.61	0.22	0.72	0.43
−0.27	−0.61	.....	.....	.....
−0.02	.....	−0.10	0.03	−0.02
−0.29	.....	0.14	−0.75	.....
−0.41	.....	−0.20	.....	−0.41
2.27	−0.78	−0.13	−0.96	−0.53
38.14	−23.22	−7.00	−1.75	−12.01
2.557	−1.191	−0.437	−0.067	−2.002

TABLE 6.—Computation of Section Corrections

Section No.	Effective length	(a) Example No. 1					(b) Example No. 2				
		Condition multipliers		Sum of Columns (3) and (4)	Section Corrections		Condition multipliers		Sum of Columns (8) and (9)	Section Corrections	
					v = Column (2) × Column (5)	Final				v = Column (2) × Column (10)	Final
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
1	3	+0.901	.....	+0.901	+ 2.7	+0.03	+1.557	−2.002	−0.445	− 1.3	−0.01
2	3	+0.901	−0.622	+0.279	+ 0.8	+0.01	+1.557	{−2.002 −0.437}	−0.882	− 2.6	−0.03
3	5	−0.622	−0.149	−0.771	− 3.9	−0.04	−0.437	+0.067	−0.370	− 1.8	−0.02
4	8	−0.622	.....	−0.622	− 5.0	−0.05	−0.437	.....	−0.437	− 3.5	−0.03
5	10	+0.901	+0.149	+1.050	+10.5	+0.10	+1.557	−0.067	+1.490	+14.9	+0.15
6	11	.....	+0.149	+0.149	+ 1.6	+0.02	−0.067	.....	−0.067	− 0.7	−0.01
7	8.5	+0.901	−0.905	−0.004	0.0	0.00	+1.557	−1.191	+0.366	+ 3.1	+0.03
8	11	.....	−0.905	−0.905	−10.0	−0.10	.....	−1.191	−1.191	−13.1	−0.13

TABLE 7.—Verification of Circuit Closures; Section Corrections

Section No.	(a) Example 1				(b) Example 2				
	Circuit No. I	Circuit No. II	Circuit No. III	Circuit No. IV	(I)	(II)	(III)	(IV)	(V)
1	+0.03	.....	.....	.....	−0.01	.....	.....	.....	−0.01
2	+0.01	.....	+0.01	.....	−0.03	.....	−0.03	.....	−0.03
3	.....	.....	−0.04	+0.04*	.....	.....	−0.02	+0.02*	.....
4	.....	.....	−0.05	.....	.....	.....	−0.03	.....	.....
5	+0.10	.....	.....	+0.10	+0.15	.....	.....	+0.15	.....
6	.....	.....	.....	+0.02	.....	+0.03	.....	−0.01	.....
7	0.00	0.00	.....	.....	+0.03	−0.13	.....	.....	.....
8	.....	−0.10	.....	.....	.....	.....	.....	.....	.....
Summation	+0.14	−0.10	−0.08	+0.16	+0.14	−0.10	−0.08	+0.16	−0.04

* This value taken in the negative direction of Section No. 3; that is, as shown by the auxiliary direction arrow in Fig. 3.

TABLE 8.—Computation of Adjusted Elevations of Bench-Marks

Section or sub-section	Observed difference in elevation	(a) Example No. 1						(b) Example No. 2					
		Correction	Adjusted difference in elevation	Initial point	Adjusted elevation	Terminal point	Adjusted elevation	Correction	Adjusted difference in elevation	Initial point	Adjusted elevation	Terminal point	Adjusted elevation
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
1	+6.46	+0.03	+6.49	*A*	416.15	*B*	422.64	−0.01	+6.45	*A*	416.15	*B*	422.60
2	+1.44	+0.01	+1.45	*B*	422.64	*C*	424.09	.....	.....	.....	.....	.....	.....
3	−2.33	−0.04	−2.37	*C*	424.09	*D*	421.72	−0.02	−2.35	*C*	424.01	*D*	421.66
5	+1.10	+0.10	+1.20	*C*	424.09	*E*	425.29	+0.15	+1.25	*C*	424.01	*E*	425.26
D to *F*	+2.46	−0.02	+2.44	*D*	421.72	*F*	424.16	−0.01	+2.45	*D*	421.66	*F*	424.11
E to *G*	+3.36	+0.01	+3.37	*E*	425.29	*G*	428.66	0.00	+3.36	*E*	425.26	*G*	428.62
E to *H*	−7.08	0.00	−7.08	*E*	425.29	*H*	418.21	+0.02	−7.06	*E*	425.26	*H*	418.20
A to *K*	−6.12	−0.07	−6.19	*A*	416.15	*K*	409.96	−0.09	−6.21	*A*	416.15	*K*	409.94

TABLE 9.—ERRORS REQUIRING DISTRIBUTION

Cycle	(a) EXAMPLE NO. 1				(b) EXAMPLE NO. 2				
	(I)	(II)	(III)	(IV)	(I)	(II)	(III)	(IV)	(V)
1.....	−14.00	+14.85	+9.71	−7.25	−14.00	+14.85	+9.71	−7.25	+5.60
2.....	− 5.51	+ 1.91	−0.72	+2.02	−11.11	+ 3.86	−2.83	+3.66	+3.25
3.....	− 1.47	+ 0.51	+0.57	+0.78	− 5.81	+ 2.02	−0.21	+2.30	+1.46
4.....	− 0.63	+ 0.22	+0.23	+0.33	− 3.18	+ 1.10	+0.10	+1.33	+0.76
5.....	.....	.....	.....	.....	− 1.77	+ 0.61	+0.10	+0.75	+0.41
	$s_4 = 1.41$; $s_3 = 3.33$; $r = 0.423$; and $\frac{r}{1-r} = 0.734$				$s_5 = 3.64$; $s_4 = 6.47$; $r = 0.562$; $\frac{r}{1-r} = 1.28$				

TABLE 10.—EXPLANATION OF ADJUSTMENT OF A LEVEL NET BY THE DISTRIBUTION METHOD

Circuit No.	Section No.	FIRST CYCLE; EXAMPLES NOS. 1 AND 2*		SECOND CYCLE			
				Example No. 1		Example No. 2	
		Closing error	Distribution	Closing error	Distribution	Closing error	Distribution
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
I	....	−14.0					
	1	.....	$\frac{3}{24.5}\times+14.00=+1.71$	.....	$\frac{3}{24.5}\times+5.51=+0.67$	−2.80	$\frac{3}{24.5}\times+11.11=+1.36$
	2	.....	$\frac{3}{24.5}\times+14.00=+1.71$	−1.82	$\frac{3}{24.5}\times+5.51=+0.67$	−1.82 −2.80	$\frac{3}{24.5}\times+11.11=+1.36$
	5	.....	$\frac{10}{24.5}\times+14.00=+5.71$	+2.79	$\frac{10}{24.5}\times+5.51=+2.24$	+2.79	$\frac{10}{24.5}\times+11.11=+4.54$
	7	.....	$\frac{8.5}{24.5}\times+14.00=+4.85$	−6.48	$\frac{8.5}{24.5}\times+5.51=+1.91$	−6.48	$\frac{8.5}{24.5}\times+11.11=+3.86$
		−14.00	+13.98	− 5.51	+5.49	−11.11	+11.12
II	....	+10.00					
	7	+ 4.85	$\frac{8.5}{19.5}\times-14.85=-6.48$	+1.91	$\frac{8.5}{19.5}\times-1.91=-0.83$	+3.86	$\frac{8.5}{19.5}\times-3.86=-1.68$
	8	.....	$\frac{11}{19.5}\times-14.85=-8.37$	.....	$\frac{11}{19.5}\times-1.91=-1.08$	.....	$\frac{11}{19.5}\times-3.86=-2.18$
		+14.85	−14.85	+1.91	−1.91	+3.86	−3.86
III	....	+ 8.00					
	2	+ 1.71	$\frac{3}{16}\times-9.71=-1.82$	+0.67	$\frac{3}{16}\times+0.72=+0.14$	+1.36 −2.80	$\frac{3}{16}\times+2.83=+0.53$
	3	.....	$\frac{5}{16}\times-9.71=-3.04$	−1.39	$\frac{5}{16}\times+0.72=+0.22$	−1.39	$\frac{5}{16}\times+2.83=+0.88$
	4	.....	$\frac{8}{16}\times-9.71=-4.85$	.....	$\frac{8}{16}\times+0.72=+0.36$	.....	$\frac{8}{16}\times+2.83=+1.41$
		+ 9.71	−9.71	−0.72	+0.72	−2.83	+2.82
IV	....	−16.00					
	3	+ 3.04	$\frac{5}{26}\times+7.25=+1.39$	−0.22	$\frac{5}{26}\times-2.02=-0.39$	−0.88	$\frac{5}{26}\times-3.66=-0.70$
	5	+ 5.71	$\frac{10}{26}\times+7.25=+2.79$	+2.24	$\frac{10}{26}\times-2.02=-0.78$	+4.54	$\frac{10}{26}\times-3.66=-1.41$
	6	.....	$\frac{11}{26}\times+7.25=+3.07$	.....	$\frac{11}{26}\times-2.02=-0.85$	.....	$\frac{11}{26}\times-3.66=-1.55$
		− 7.25	+7.25	+2.02	−2.02	+3.66	−3.66
V*	....	+ 4.00					
	1	+ 1.71	$\frac{3}{6}\times-5.60=-2.80$	.....	.....	+1.36	$\frac{3}{6}\times-3.25=-1.63$
	2	+1.71 −1.82	$\frac{3}{6}\times-5.60=-2.80$	.....	.....	+1.36 +0.53	$\frac{3}{6}\times-3.25=-1.62$
		+5.60	−5.60	.....	.....	+3.25	−3.25

* Circuit No. V applies to Example No. 2 only.

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PAPERS

STRUCTURAL BEAMS IN TORSION

BY INGE LYSE,[1] M. AM. SOC. C. E.,
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SYNOPSIS

Results of a study of the torsional properties of standard structural steel beams are offered for discussion in this paper. The purpose of the investigation was to furnish a reliable basis for the design of structural members subjected to torsional loads. The relation between torque and stress on the one hand, and between torque and twist on the other, for any piece subjected to torsion involves a constant the value of which is a function of the material and the shape of the cross-section. An accurate method is given for the evaluation of this torsion constant, K, for standard H-sections and I-sections, taking full account of all factors involved. This has been made possible by applying the "membrane analogy" to about sixty sections of widely varying flange, web, and fillet proportions.

The investigation included a study of the effect of end fixity in torsional design, and shows how it may be obtained effectively. The proposed formulas are applied to practical design problems, and are checked by torsional tests on structural steel sections ranging in size from a 3-in. I-beam weighing 7.5 lb per ft, to a 12 by 12-in. beam weighing 190 lb per ft.

HISTORICAL FOREWORD

The problem of pure torsion as applied to non-circular sections was first treated correctly by Saint Venant (1)[3] in 1855, and his general solution is applicable to any cross-section. In 1903, Prandtl (2) showed that if a thin membrane were stretched across a hole having the shape of the cross-section in question and distorted slightly, the equation of its surface had the same form as the general differential equation involved in the torsion problem.

NOTE.—Discussion on this paper will be closed in August, 1935, *Proceedings*.

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[3] The numbers in parentheses refer to references given in Appendix I.

Prandtl showed that by measuring the volume and slopes of the displaced membrane a direct measurement of the torsional rigidity and stress was obtainable. Prandtl's analogy, with a thin soap film as a membrane, was used in several torsion investigations, first in England by Griffith and Taylor (3) who studied the torsional strength of aeroplane sections in 1917, and, later, in the United States by Trayer and March (4), who, in 1930, made similar studies for the same purpose.

Important contributions to the torsion problem have been made by Timoshenko (5). He has shortened the pure torsional theory by slight modifications of Saint Venant's equations and by mathematical application of the principles of the membrane analogy. He was also among the first to consider the effect produced by preventing the warping of a cross-section. This problem has had the attention of numerous investigators in connection with problems of elastic stability and buckling during bending. Sonntag (7) treated the theoretical aspects of this problem in an article published in 1929.

Introduction

The investigation reported herein was undertaken as a study of all available information on the subject, both theoretical and experimental, supplemented by a considerable number of actual torsion tests of structural steel beams and soap film experiments on various cross-sectional shapes. The writers first considered testing beam sections 3 ft in length, welded to thick plates at the ends. A study of the problem, however, showed that such beams would be several times stronger than if they were tested free-ended, and that, unless the exact percentage of end fixity was known, it would not be possible to draw definite conclusions from such tests.

In order to study the effect of end fixity directly, tests were first made on eight sections of a 3-in. I-beam (7.5 lb per ft), varying from 3 in. to 4 ft 6 in. in length and cut from the same rolled section. The ends of each piece were welded to plates 1 in. thick, and the specimens were tested in a standard 26 000 in-lb torsion machine.

The results of these tests pointed the way to a revised general program, and a torsion testing rig capable of applying torsional load up to 750 000 in-lb was designed. Provision was made for testing beams either fixed or free at the ends, and with lengths of 1 ft 6 in., 3 ft, or 6 ft. Nineteen different tests were made, twelve on beams with the ends fixed by welding the side and end plates to form a box section at the ends, and seven with ends free. The beams ranged in sizes from the 3-in. I-beam, weighing 7.5 lb per ft, to a 12 by 12-in. beam weighing 190 lb per ft. Tensile and shearing properties of the material in each type of beam were obtained by standard tensile tests, round bar torsion tests, and slotted plate shear tests. Soap film experiments on fifty-seven differently proportioned sections were made for the determination of the torsional rigidity.

This paper contains the final summary of all phases of the investigation. Use has been freely made of the findings of previous investigators, for which acknowledgment is made at appropriate points.

Notation.—The symbols in this paper are introduced in the text as they occur, and are summarized for reference in Appendix II.

The Torsion Theory

General Problem.—The solution of the torsional properties of a section of any shape consists primarily in determining the distribution of lateral shearing stresses over the cross-section. The shear components will be of uneven distribution, except in the case of the circular section, and as a result plane sections will be warped during twisting as shown in Fig. 1. (Note that, in Fig. 1(*c*), the web and each flange are warped as individual rectangles, in addition to the warping of the section as a whole.)

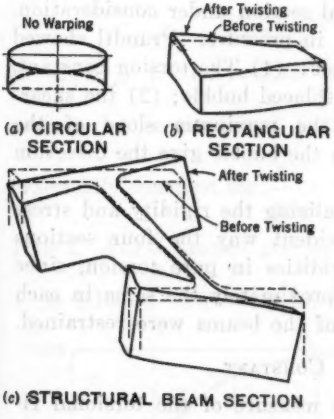

FIG. 1.—TWISTING OF BARS OF VARIOUS CROSS-SECTIONS

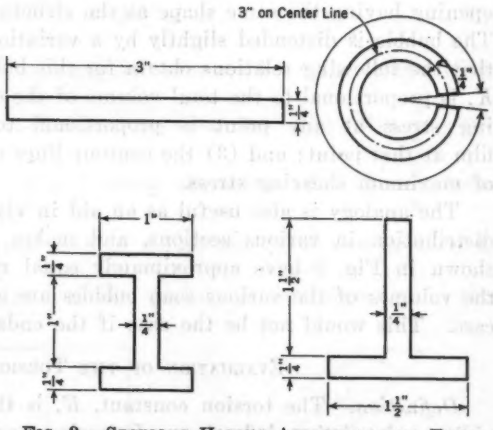

FIG. 2.—SECTIONS HAVING APPROXIMATELY EQUAL TORSIONAL RIGIDITY

It is assumed that the lateral displacements are proportional to the angular twist and to the distance from the twisting axis (as is the case in a circular section). The longitudinal displacements cause the warping, and the resulting distribution of shearing stress is taken care of by introducing a "stress-function," F, of x and y. This function must satisfy the differential equation:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = -2G\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which, G = the shearing modulus of elasticity, and θ = angle of twist, in radians per inch. It may be shown that the function, F, must be a constant along the boundary of the section for solid bars, and, therefore, may be chosen arbitrarily as equal to zero.

If the boundary conditions are such that Equation (1) may be solved and the value of F determined, it is possible to evaluate the torsion constant of the section and find the stress at any point in the cross-section. Formulas for the torsion constant and critical shearing stresses have been derived in this manner for such sections as the square, rectangle, ellipse, equilateral triangle, and sector of a circle (1).

In the case of the circular shaft the shearing stress components have a uniform distribution along each radius, and since the longitudinal shear is likewise evenly distributed, there is no longitudinal warping of first-order importance. The well-known simple theory using the polar moment of inertia is thus applicable to the case of the circular section.

If, in some way, warping which takes place in non-circular sections is restrained or prevented, longitudinal fiber stresses will be introduced and the beam stiffened and strengthened.

The Membrane Analogy.[4]—Equation (1) may be solved mechanically for any cross-section by means of Prandtl's membrane analogy, thereby overcoming the mathematical limitations of the theoretical derivation.

In the application of this analogy, a soap film is stretched across an opening having the same shape as the structural section under consideration. The bubble is distended slightly by a variation in pressure. Prandtl showed that the following relations obtain for this bubble: (1) The torsion constant, K, is proportional to the total volume of the displaced bubble; (2) the shearing stress at any point is proportional to the maximum slope of the film at that point; and (3) the contour lines on the bubble give the direction of maximum shearing stress.

The analogy is also useful as an aid in visualizing the rigidity and stress distribution in various sections, and makes evident why the four sections shown in Fig. 2 have approximately equal rigidities in pure torsion, since the volumes of the various soap bubbles are approximately the same in each case. This would not be the case if the ends of the beams were restrained.

Evaluation of the Torsion Constant

Definition.—The torsion constant, K, is the measure of the torsional rigidity and twisting deflections. It is also a part of any formula for torsional shearing stresses, and may be determined from test results by observing the ratio of torsional moment to unit twist, in radians per inch, at any place below the yield point of the beam, and dividing this ratio by the shearing modulus of elasticity.

The Relation Between K and J.—When a torsional couple, T, is applied to a circular shaft of radius, r, the maximum shearing stress, τ, at the surface, is given by:

$$\tau = \frac{T r}{J} \qquad (2)$$

in which, J = polar moment of inertia. In terms of T Equation (2) may be re-arranged to read:

$$T = \frac{\tau J}{r} \qquad (3)$$

The torque, T, may also be expressed in terms of θ and G, thus:

$$T = J G \theta \qquad (4)$$

[4] A detailed description of the soap film studies is given in a thesis by Bruce G. Johnston, Jun. Am. Soc. C. E., presented to Lehigh University in partial fulfillment of the requirements for the degree of Master of Science.

For non-circular sections the torsional resisting moment may again be expressed in terms of θ and G, with the substitution of K, the torsion constant, in place of J, thus:

$$T = K G \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The torsion constant, K, is equal to the polar moment of inertia for circular sections. Although for non-circular sections it is always less than the polar moment of inertia, there is no direct relation between the two factors.

The Rectangle.—In dealing with structural shapes, two principal types of section require consideration, the rectangle, and the rectangle modified by sloping sides, as in the flange of a standard I-beam. In the case of the rectangle an accurate formula was derived originally by Saint Venant (1):

$$K = \frac{n^3 b}{3} - 2 V n^4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

in which n = the breadth of a rectangular section; b = the length of a rectangular section; and V = a factor depending upon the ratio, $\frac{b}{n}$, but practically constant for $\frac{b}{n} > 3$. Fig. 3 shows the values of V for $\frac{b}{n}$ from 1 to 3. For $\frac{b}{n}$ -ratios greater than 3, $V = 0.105$, and for $\frac{b}{n}$ greater than 4, $V = 0.10504$. Equation (6) finds a direct, qualitative interpretation in the

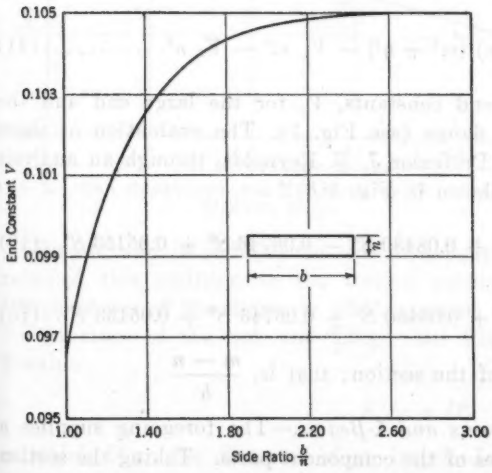

FIG. 3.—END CONSTANTS FOR RECTANGULAR SECTIONS, WITH $\frac{b}{n} < 3$. (SEE EQUATION (6)).

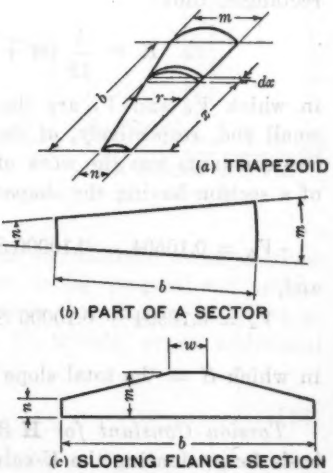

FIG. 4.

soap film analogy. It is evident that for long rectangular sections the bubble will be of constant cross-section along the central part, but at the two ends it will be contracted and brought down to meet the small side. The

quantity, $-2\ V\ n^4$, then represents the "end loss," which for long sections is evidently a function of n only.

It also follows that if the ends were made discontinuous, as if they were parts of infinitely long rectangles, one might state without error:

$$K = \frac{1}{3}\ n^3\ b \qquad \text{.....(7)}$$

and for any differential length, dx, along the section:

$$K = \frac{1}{3}\ n^3\ dx \qquad \text{.....(8)}$$

The Section with Sloping Sides.—Equation (8) provides a basis for evaluating K for the sloping flange section. Considering the section shown in Fig. 4(a), let the thickness at any point be taken as r. Then, if the ends are assumed discontinuous:

$$K = \frac{1}{3}\int_0^b r^3\ dx \qquad \text{.....(9)}$$

Evaluating r in terms of m and n, and integrating:

$$K = \frac{b}{12}\ (m + n)\ (m^2 + n^2) \qquad \text{.....(10)}$$

in which m = major flange thickness and n = minor flange thickness. A deduction must be made for end effects, as in the case of the simple rectangle, thus:

$$K = \frac{b}{12}\ (m + n)\ (m^2 + n^2) - V_L\ m^4 - V_s\ n^4 \qquad \text{.....(11)}$$

in which V_L and V_s are the end constants, V, for the large end and the small end, respectively, of the flange (see Fig. 5). The evaluation of these two constants was the work of Professor J. B. Reynolds, through an analysis of a section having the shape shown in Fig. 4(b):[5]

$$V_L = 0.10504 - 0.10000\ S + 0.08480\ S^2 - 0.06746\ S^3 + 0.05153\ S^4 \qquad \text{..(12)}$$

and,

$$V_s = 0.10504 + 0.10000\ S + 0.08480\ S^2 + 0.06746\ S^3 + 0.05153\ S^4 \qquad \text{..(13)}$$

in which S = the total slope of the section; that is, $\dfrac{m - n}{b}$.

Torsion Constant for **H***-Beams and* **I***-Beams.*—The foregoing supplies a basis for evaluating the K-values of the component parts. Taking the section shown in Fig. 4(c) as a basis for the sloping flange section, the sum of two trapezoids and a small rectangular part is expressed by:

$$K_f = \frac{b - w}{12}\ (m + n)\ (m^2 + n^2) + \frac{1}{3}\ w\ m^3 - 2\ V_s\ n^4 \qquad \text{.....(14)}$$

[5] "Theory of Elasticity," by A. E. H. Love, Fourth Edition, p. 319.

in which K_f = the K-value for the flange. The web is considered as a discontinuous section between the flanges, giving:

$$K_w = \frac{1}{3}\ (d - 2\,m)\ w^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

in which K_w = the K-value for the web; d = total depth of beam; and, w = thickness of web (see Fig. 6). There still remains the evaluation of the added rigidity due to the connection of the flange and web and also due to the fillet at this point. It is evident that these will cause a considerable "hump" in the soap bubble.

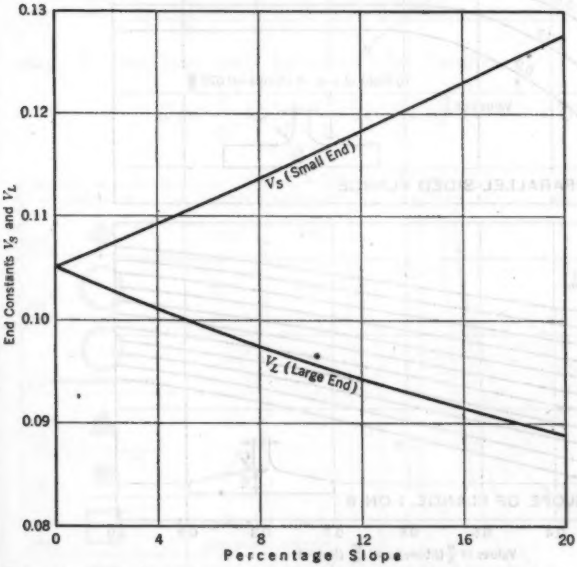

FIG. 5.—END CONSTANTS FOR K-VALUES OF FLANGES WITH SLOPING SIDES.

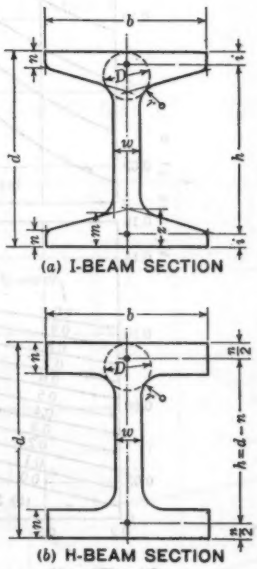

FIG. 6.

Trayer and March (4) in an investigation of aircraft strut sections assumed this addition to the torsion constant to be proportional to the fourth power of the diameter of the largest circle that can be inscribed at the juncture of the web and flange (see Fig. 6), giving, as an additional K-value:

$$K = \alpha\ D^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

in which D = the diameter of an inscribed circle; and α = a factor that depends on two ratios, $\frac{w}{m}$ and $\frac{r}{m}$.

Values of α for sections with parallel-sided flanges, and for sections with flange slopes of 1 on 6, are given in Fig. 7. These curves were determined experimentally as the result of soap film tests which will be described subsequently. It will be noted in Fig. 7(*a*) that for parts of the curves to the

right of *a-a*, the lines are parallel and uniformly spaced. All standard rolled beams are in this area, in which case, for parallel flange sides:

$$\alpha = 0.094 + 0.070 \frac{r}{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

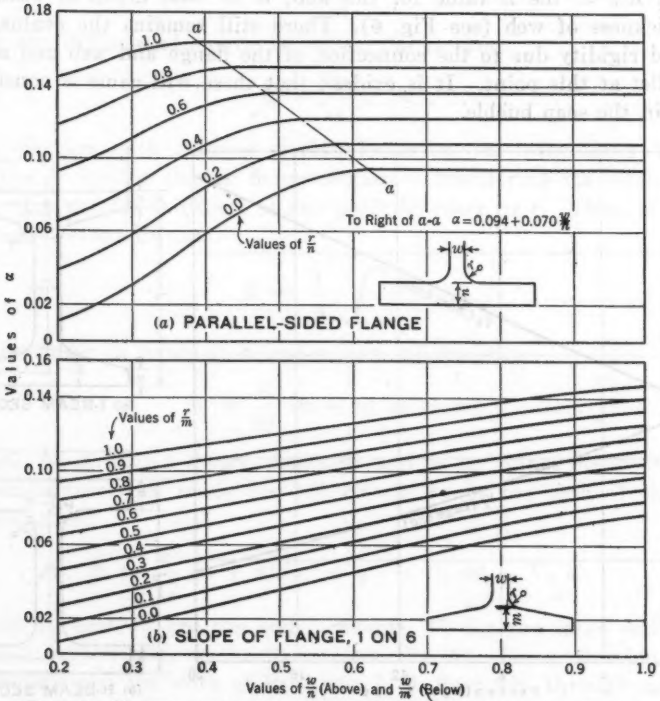
Values of α
(a) PARALLEL-SIDED FLANGE
To Right of a-a α=0.094+0.070 r/m
Values of r/m
(b) SLOPE OF FLANGE, 1 ON 6
Values of w/m (Above) and w/m (Below)

FIG. 7.—VALUES OF α FOR RIGHT-ANGLE JUNCTION OF FLANGE AND WEB.

For flange sections with side slopes of 1 on 20 and 1 on 50 the following formulas represent an interpolation between the curves in Fig. 7:

For a slope of 1 on 20:

$$\alpha = 0.066 + 0.021 \frac{w}{m} + 0.072 \frac{r}{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

and, for a slope of 1 on 50:

$$\alpha = 0.084 + 0.007 \frac{w}{m} + 0.071 \frac{r}{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

The various elements entering into the total K-values, can now be summarized as follows (refer to Fig. 6):

For sloping flange sections,

$$K = \frac{b - w}{6} (m + n) (m^2 + n^2) + \frac{1}{3} (d - 2m) w^3 + 2 \alpha D^4 - 4 V_s n^4 \ldots (20)$$

Values of V_L and V_S for standard slopes are, as follows:

S	V_L	V_S
$\frac{1}{6}$	0.09045	0.12441
$\frac{1}{20}$	0.10026	0.11026
$\frac{1}{50}$	0.10307	0.10707
$\frac{1}{\infty}$	0.10504	0.10504

and, for sections with parallel-sided flanges,

$$K = \frac{2}{3} b n^3 + \frac{1}{3} (d - 2n) w^3 + 2 \alpha D^4 - 0.42016 n^4 \quad \ldots\ldots (21)$$

SECTIONS	RIGIDITY	STRENGTH
(a)	100.0	100.0
(b)	637.0	332.0
(c)	5.5	18.0
(d)	70.0	62.0
(e)	88.0	74.0
(f)	341.0 (Approx.)	280.0 (Approx.)
(g)	9.9 (Nearly Exact)	22.2 (Approx.)
(h) $r = w = \frac{2}{3} n$	11.6 (Nearly Exact)	22.8 (Approx.)
(i) $6\frac{1}{2}$" Fixed Ends 1'6"	78.1 (Approx.)	38.3 (Approx.)

FIG. 8.—TORSIONAL RIGIDITY AND STRENGTH OF DIFFERENT SECTIONS OF EQUAL CROSS-SECTION AREAS.

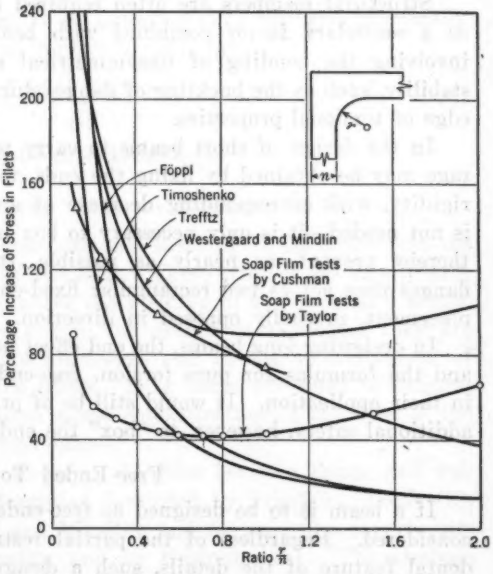

FIG. 9.—FORMULAS FOR STRESS CONCENTRATION IN FILLETS DUE TO TORSION.

Diameter D may be determined by a large scale layout, or by the following formulas:

For parallel-sided flange sections,

$$D = \frac{(n + r)^2 + w\left(r + \frac{w}{4}\right)}{2r + n} \quad \ldots\ldots (22)$$

for sloping-sided flange sections,

$$D = \frac{(B+z)^2 + w\left(r + \frac{w}{4}\right)}{B + r + z} \quad \ldots\ldots\ldots\ldots\ldots (23)$$

in which z = the maximum flange depth shown in Fig. 6, and,

$$B = r\,S\left[\sqrt{\frac{1}{S^2} + 1} - 1 - \frac{w}{2\,r}\right] \quad \ldots\ldots\ldots\ldots\ldots (24)$$

Comparative Efficiencies of Different Sections.—Fig. 8 shows the comparative torsional efficiencies of certain different shapes, illustrating the striking advantages of the hollow box, or tubular, construction. These advantages would not obtain entirely if the section were built up by use of bolts or rivets.

Examples of Design

General Statement

Structural members are often required to carry torsional loads, generally as a secondary factor combined with bending or direct stress. Problems involving the bending of unsymmetrical sections, and problems of elastic stability, such as the buckling of flanges during bending, also require a knowledge of torsional properties.

In the design of short beams to carry torsional loads considerable advantage may be obtained by fixing the ends, resulting in increased strength and rigidity, with corresponding decrease of angular deflection. External fixity is not needed—it is only necessary to box in the two flanges at each end and thereby prevent, as nearly as possible, their relative warping. The two flanges then act as two rectangular fixed-ended beams carrying a lateral displacement, mutually opposed in direction.

In designing long beams, the end effect tapers out rapidly toward the center, and the formulas for pure torsion, free-ended, will be adequate and simpler in their application. It would still be of practical advantage and a means of additional safety, however, to "box" the ends of the beam.

Free-Ended Torsion

If a beam is to be designed as free-ended, only shearing stresses need be considered. Regardless of the partial restraint that does exist as an incidental feature of the details, such a design will be on the safe side. The critical shearing stresses will occur along the outer surface of the beam where the material is thickest, generally along the outside center line of the flange and along the inside re-entry fillets.

The shearing stress is a function of the thickness of the material and the following formula is proposed for the maximum shearing stress in the flange of an H-beam or an I-beam in free-ended torsion:

For parallel-sided flange sections,

$$\tau_f = \frac{T\,(D+n)}{2\,K} \quad \ldots\ldots\ldots\ldots\ldots (25a)$$

for sloping flange sections,

$$\tau_f = \frac{T\,(D + m)}{2\,K} \qquad \text{.....(25b)}$$

and for shear stress in the web,

$$\tau_w = \frac{T_w}{K} \qquad \text{.....(26)}$$

in which T = torsional moment. Although Equation (26) is in accord with torsional theory it gave results which proved low by comparison with actual tests. The following tentative formula was found to give better agreement:

$$\tau_w = \frac{T\,(w + 0.3\,r)}{2\,K} \qquad \text{.....(27)}$$

Equation (27) was adopted for use in the reported investigation. More accurate stress formulas might be developed by further use of soap film tests, taking slope measurements at critical points, and testing sections with various ratios of web, fillet, and flange as was done in determining the torsion constant. An equation similar to Equation (27) will give practically the same values for flange stress as Equation (25), thus:

$$\tau_f = \frac{T\,(n + 0.3\,r)}{K} \qquad \text{.....(28)}$$

Equation (25) was used in the reported investigation.

Concentration of Shearing Stress at Re-Entry Fillet

At the re-entry fillet of an I-beam torsional shearing stress concentrations occur due to the sharp curvature of the fillet. These stresses are mostly of a local nature and do not greatly influence the yielding of the beam as a whole. Although they do not necessarily govern the design of the beam they are important in the study of adequate fillet sizes and in the determination of loads producing strain lines in the fillets. In a beam subjected to flexural as well as torsional stresses the total stress concentration at the fillet is the sum of the stresses due to these two causes.

The concentration of torsional stress at the fillet between flange and web is illustrated in Fig. 9 for the formulas developed by Föppl(9), Trefftz(8), Timoshenko(5), and Westergaard and Mindlin, and for soap film experiments conducted by Griffith and Taylor(3), and by Cushman(13). The analysis resulting in the Westergaard-Mindlin curve was communicated to the writers by its originators. It is noted that all curves indicate a rapid increase in stress concentration with the decrease of the ratio, $\frac{r}{n}$, between the radius of the fillet and the thickness of the section next to it, particularly for ratios of less than 1.0. Strain lines, therefore, will appear at relatively small torsional moment for sections having small fillets.

As the fillets become increasingly large another factor is introduced since the increased stress due to the greater thickness of material becomes of more

importance. The rigidity of the beam is increased in proportion to the third power of the thickness, whereas the stress varies directly with the thickness. Hence, for any given moment the stress would actually decrease. The curve of soap film tests by Griffith and Taylor indicate this reverse effect, but further experiments along this line should be conducted to establish these relations more definitely.

Fixed-Ended Torsion

Assumptions.—If the ends of a non-circular section are fixed in some manner so as to prevent free warping of the end section, pure torsion no longer exists.

Various investigators have studied this problem—Föppl(9), Timoshenko(5), Sonntag(7), and others. In 1930 an investigation on channel sections was reported by Seely, Putnam, and Schwalbe (10). These investigators have generally been concerned with the problem of elastic equilibrium involved in the side buckling and twisting of a beam in bending without lateral support. Hence the torsion problem has been a secondary issue.

In the present tests both ends of structural beams have been fixed by welding on heavy end plates and additional side stiffening plates between the flanges at each end of the beam. The major stiffening effect produced is that of fixing the flanges relative to each other. (Refer to Fig. 10.) The

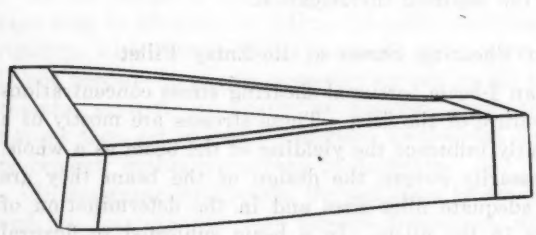

FIG. 10.—BEAM WITH FIXED ENDS.

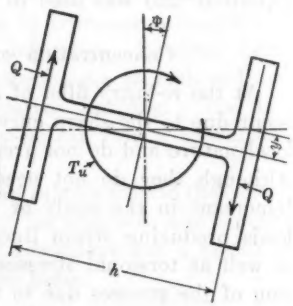

FIG. 11.—TWISTING OF A STRUCTURAL SHAPE.

prevention of the individual warping of the component rectangular parts would taper out so rapidly that it would be of negligible importance; but, by fixing the flanges with respect to each other, the effect of two opposed fixed-ended beams is produced, as illustrated in Fig. 10.

The following assumptions have been made: (1) The flanges remain at right angles to the web; (2) the angular deflection is small compared with the length of the beam; (3) the bending of each flange about its weaker axis is a negligible factor; (4) a point on the neutral axis of one flange can be located with sufficient accuracy by co-ordinates (x, y) measured along the perpendicular to the original position of the axes; (5) the two ends of the beam are held between mutually parallel planes as twisting takes place; (6) the

displacement of the flanges due to beam action is due to bending only (that is, lateral shearing deflection is neglected; a correction is made afterward for shearing deflection in very short beams); and (7) $\frac{I_y}{2}$ = moment of inertia of one flange about the web axis.

Assumption (7) is quite accurate and of great convenience in the case of standard **I**-beams and **H**-beams for which I_y is given in all handbooks. The following symbols in addition to those previously given will be used:

$$a = \frac{h}{2}\sqrt{\frac{EI_y}{KG}}$$

or, for steel,

$$a = 0.806\ h\ \sqrt{\frac{I_y}{K}}$$

and for sloping flange sections,

$$h = d - \frac{2}{3}\left(n + \frac{z^2}{z + n}\right)$$

in which h = distance between flange centroids; or, approximately,

$$h = d - \frac{m + n}{2}$$

which is sufficiently accurate for practical purposes.

Derivation of General Equations.—Consider the equilibrium of any cut section as shown in Fig. 11. The outer torsional moment must be resisted by the internal moment of the resisting forces, thus:

$$T = T_u + Qh \qquad (29)$$

in which T_u = torque required to twist a beam in a free-ended condition; and Q = total lateral shearing force developed by one flange. In terms of the chosen co-ordinates, $\frac{h}{2}\ d\psi = dy$, approximately, and the twist, ψ, per unit length $= \frac{d\psi}{dx}$. Hence,

$$T_u = KG\frac{d\psi}{dx} = \frac{2KG}{h}\frac{dy}{dx} \qquad (30)$$

Furthermore, assuming that the larger moment of inertia of one flange is equal to $\frac{I_y}{2}$,

$$\frac{EI_y}{2}\frac{d^3y}{dx^3} = -Q \qquad (31)$$

Considering Equations (29), (30), and (31):

$$\frac{EI_y}{2}\frac{d^3y}{dx^3} - \frac{2KG}{h^2}\frac{dy}{dx} = \frac{-T}{h} \qquad (32)$$

Differentiating with respect to x, and substituting $a = \frac{h}{2}\sqrt{\frac{EI_y}{KG}}$:

$$a^2 \frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 0 \quad \ldots\ldots(33)$$

As a general solution of this differential equation:

$$y = A \sinh \frac{x}{a} + B \cosh \frac{x}{a} + C + Dx \quad \ldots\ldots(34)$$

Torsion with Both Ends Restrained.—By proper evalution of the constants for the conditions obtaining in a beam fixed at both ends:

$$y = \frac{T\,h\,a}{2\,K\,G}\left(\cosh \frac{x}{a} \tanh \frac{l}{2\,a} - \sinh \frac{x}{a} + \frac{x}{a} - \tanh \frac{l}{2\,a}\right) \ldots(35)$$

in which l = length of the beam. For $x = l$, the total deflection equals,

$$y = \frac{T\,h\,a}{2\,K\,G}\left(\frac{l}{a} - 2 \tanh \frac{l}{2\,a}\right) \quad \ldots\ldots(36)$$

The moment in each flange equals,

$$M = \frac{E\,I_y}{2}\frac{d^2y}{dx^2} = \frac{T\,a}{h}\frac{\sinh u}{\cosh \frac{l}{2\,a}} \quad \ldots\ldots(37)$$

in which $u = \frac{l}{2\,a} - \frac{x}{a}$; and, furthermore, the shear in each flange equals,

$$Q = \frac{E\,I_y}{2}\frac{d^3y}{dx^3} = \frac{-T}{h}\frac{\cosh u}{\cosh \frac{l}{2\,a}} \quad \ldots\ldots(38)$$

The longitudinal stresses along the outer fibers of the flanges will be given by:

$$\sigma = \frac{M\,c}{I} = \frac{M\,b}{I_y} = \frac{T\,a\,b}{h\,I_y}\frac{\sinh u}{\cosh \frac{l}{2\,a}} \quad \ldots\ldots(39)$$

Critical stress will be either longitudinal stresses at the end of the beam, or the sum of the lateral and torsional shearing stresses at the center of the beam.

At the end of the beam, $u = \frac{l}{2\,a}$, and, consequently,

$$\sigma = \frac{T\,a\,b}{h\,I_y} \tanh \frac{l}{2\,a} \quad \ldots\ldots(40)$$

and at the center of the beam, $u = 0$, giving,

$$Q_c = \frac{T}{h} \operatorname{sech} \frac{l}{2\,a} = \frac{T}{h \cosh \frac{l}{2\,a}} \quad \ldots\ldots(41)$$

At the center of the beam both the lateral and torsional shearing stresses will be maximum in the middle of the flange.

By the simple beam theory, the lateral shearing stress will be: For parallel flange sections,

$$\tau = \left(\frac{3}{2}\,\frac{Q}{A_f}\right) = \frac{Q\,b^2}{4\,I_y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

in which $A_f = A$ for one flange. For sloping flange sections,

$$\tau = \frac{Q\,b^2\,(2\,n + m)}{12\,m\,I_y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43)$$

For torsional shearing stress at the center of the beam, the approximate formulas proposed in Equations (25) and (27) are applied. Instead of K, the torsion constant, an equivalent constant must be introduced, which is measured by the rate of angular twist at the center of the beam.

To obtain this, differentiate Equation (35) with respect to x, giving the slope of the flange, $\frac{dy}{dx}$, at any point. The unit angular twist, $\frac{d\psi}{dx} = \frac{2}{h}\,\frac{dy}{dx}$, and from this relation and the substitution of $x = \frac{l}{2}$, an expression is derived for $\theta_c = \frac{d\psi}{dx}$ at the center; thus,

$$\theta_c = \frac{T}{K\,G}\left[\frac{\cosh\left(\frac{l}{2\,a}\right) - 1}{\cosh\left(\frac{l}{2\,a}\right)}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (44)$$

This derivation (Equation (44)) has been based on the assumption (6) that deflection is due to bending only. As the beam is shortened, however, shearing deflection becomes of increasing importance and should be considered.

Timoshenko(6) has indicated a strain energy method for calculating the deflection due to shear when cross-sections are constrained from warping. By combining his result for the simple cantilever beam the correction, $1 + 2.95\,\frac{b^2}{l^2}$, is obtained for a fixed-ended beam with point of inflection at the center; that is,

$$\theta_c = \frac{T}{K\,G}\left[\frac{\cosh\left(\frac{l}{2\,a}\right) - 1}{\cosh\left(\frac{l}{2\,a}\right)}\right]\left(1 + 2.95\,\frac{b^2}{l^2}\right) \quad \ldots\ldots\ldots (45)$$

and, denoting the "equivalent" torsion constant at the center by C_c:

$$C_c = \frac{T}{\theta_c\,G} = K\left[\frac{\cosh\left(\frac{l}{2\,a}\right)}{\cosh\left(\frac{l}{2\,a}\right) - 1}\right]\left[\frac{1}{1 + 2.95\,\frac{b^2}{l^2}}\right] \quad \ldots\ldots\ldots (46)$$

Using C_c as the measure of torsional shear developed at the center of the beam instead of K and combining Equations (25) with Equations (42) and (41), Equations (47) to (53) are derived for combined torsional and lateral shearing stresses, as follows:

Total Maximum Shearing Stress at Center of Beam (Along Center Line of Flange).—For parallel flange sections:

$$\tau = T\left[\frac{b^2}{4\,h\,I_y \cosh\left(\frac{l}{2\,a}\right)} + \frac{(D+n)}{2\,C_c}\right] \quad \ldots\ldots\ldots (47a)$$

for sloping flange sections:

$$\tau = T\left[\frac{b^2\,(2\,n+m)}{12\,h\,m\,I_y \cosh\left(\frac{l}{2\,a}\right)} + \frac{D+m}{2\,C_c}\right] \quad \ldots\ldots\ldots (47b)$$

and the stress in the web may be computed by:

$$\tau_w = \frac{T\,(w+0.3\,r)}{C_c} \quad \ldots\ldots\ldots (48)$$

Total Twisting Deflections of Fixed-Ended Beams.—Equation (46) provides an equivalent torsion constant based on the unit angular twist at the center of the beam. A measure of the total twisting deflection of the beam over the entire length is desired and can be obtained effectively by evaluating an average equivalent torsion constant, which will be denoted as C_A.

The expression for total angular twist is, then:

$$\psi = \frac{T\,l}{C_A\,G} \quad \ldots\ldots\ldots (49)$$

For very short beams most of the deflection is that producing shear and C_c approaches C_A in value. Equation (36) is an expression for the total deflection of the flanges due to bending only. Constant C_A may be evaluated from Equation (36) in so far as bending deflections are concerned.

The ratio of $\frac{C_A}{C_c}$ reduces to the following expression:

$$\frac{C_A}{C_c} = \frac{\left(\cosh\frac{l}{2\,a} - 1\right)}{\left(\cosh\frac{l}{2\,a}\right) - \left(\frac{2\,a}{l}\sinh\frac{l}{2\,a}\right)} \quad \ldots\ldots\ldots (50)$$

The graph of $\frac{C_A}{C_c}$ is given on Fig. 12, permitting the quick calculation of C_A after C_c is known. As the length approaches zero, C_A should approach C_c in value and the curve on Fig. 13 gives the reduction to be made in C_A for various $\frac{l}{b}$-ratios in terms of a factor to be multiplied by $(C_A - C_c)$.

As an example, let: $C_c = 6.00$ by Equation (46); $\frac{l}{2\,a} = 2.00$; and, $\frac{l}{b} = 5.00$.

From Fig. 12, $\frac{C_A}{C_c} = 1.42$; and, from Fig. 13, the reduction $= (0.42)\ (0.105)$ $= 0.44$; or, $C_A = 6.00\ (1.42 - 0.044) = 8.28$.

Torsion with One End Fixed and One End Free.—Often, in cases of combined bending and torsion, one end of the beam will be relatively unre-

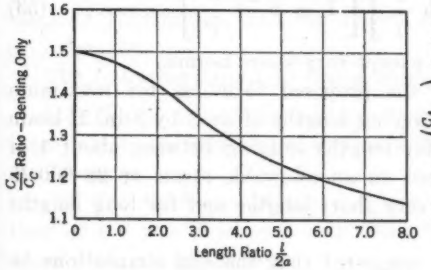

FIG. 12.—POSITIVE FACTOR TO OBTAIN C_c. TO BE USED IN CONJUNCTION WITH NEGATIVE FACTOR FROM FIG. 13.

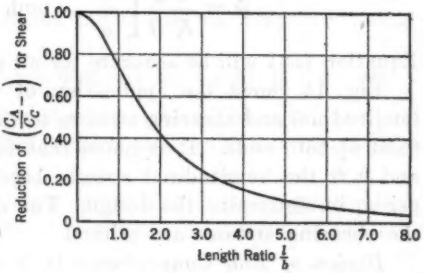

FIG. 13.—REDUCTION FACTOR FOR SHEAR (FOR USE WITH FIG. 12 TO OBTAIN C_A FROM C_c).

strained while the other end is fixed. At the free end there will be no lateral shearing stresses in the flanges and the shearing stress formulas (Equations (25)) for free end torsion will apply. The evaluation of Equation (34) for these end conditions gives the following: For maximum bending moment in the flange at the unrestrained end:

$$M_{max} = \frac{T\,a}{h} \tanh \frac{l}{a} \qquad \text{.....(51)}$$

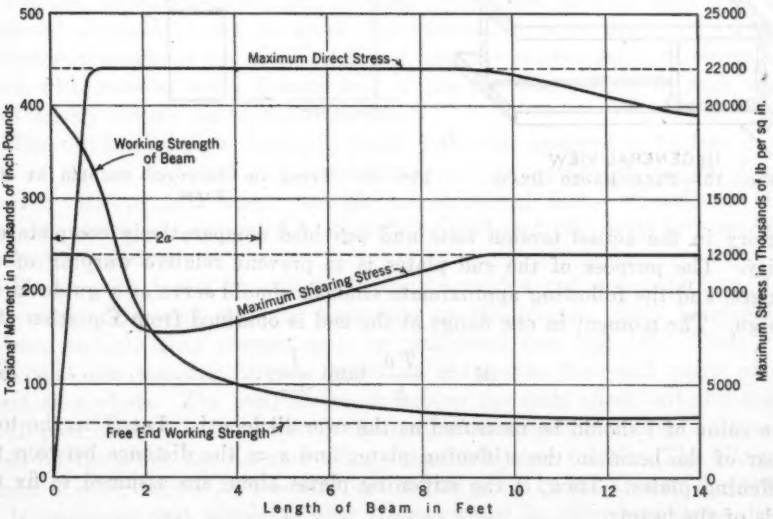

FIG. 14.—WORKING STRENGTH OF AN 8 BY 8-INCH BEAM WITH FIXED ENDS (SHOWING LIMITING STRESSES FOR DIFFERENT LENGTHS).

and for maximum direct fiber stress in the outer edges of the flanges at the restrained end:

$$\sigma = \frac{T\,a\,b}{h\,I_y} \tanh \frac{l}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52)$$

The total angle of twist is given by:

$$\psi = \frac{T\,a}{K\,G}\left[\frac{l}{a} - \tanh \frac{l}{a}\right]\left[1 + 0.74\,\frac{b^2}{l^2}\right] \quad \ldots\ldots\ldots\ldots (53)$$

Equation (53) will be accurate for all except very short beams.

Fig. 14 shows the application of the proposed formulas for maximum longitudinal and shearing stresses to varying lengths of an 8 by 8-in. H-beam fixed at both ends. It is noted that for lengths ranging between about 1 ft and 9 ft the longitudinal stresses based on an allowable stress of 22 000 lb per sq in. determine the design. For very short lengths and for long lengths the shearing stresses are critical.

Design of End Connection.—It is suggested that the end connections be built as illustrated in Fig. 15. Connections of this type proved very satis-

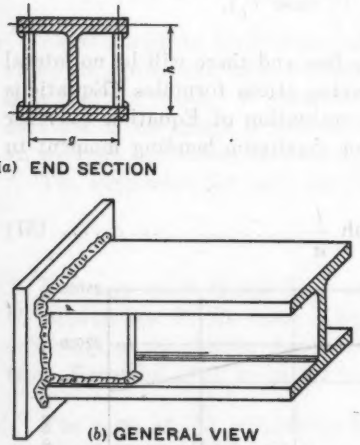

FIG. 15.—FIXED-ENDED BEAM.

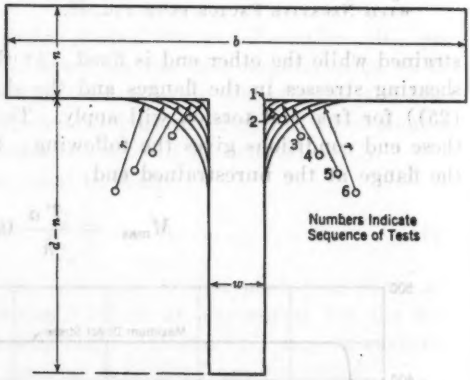

FIG. 16.—TYPES OF SECTIONS STUDIED BY SOAP FILM.

factory in the actual torsion tests and provided comparatively complete end fixity. The purpose of the end plates is to prevent relative warping of the flanges, and the following approximate analysis should serve as a guide in the design. The moment in one flange at the end is obtained from Equation (37)

$$M = \frac{T\,a}{h} \tanh \frac{l}{2\,a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54)$$

The value of l should be measured as the over-all length. Let Q = the total shear of the beam in the stiffening plate; and s = the distance between the stiffening plates. Then, if the stiffening plates alone are assumed to fix the ends of the beam:

$$M = Q\,s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55)$$

Substituting in Equation (54):

$$Q = \frac{T a}{h s} \tanh \frac{l}{2 a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56)$$

The stiffener plate is welded to both flanges and to the end plate as well. The stiffener and the adjoining part of the end plate act as a short fixed-ended beam holding the flanges in place. No attempt was made to analyze the load distribution, and the design of the test beams was largely a matter of judgment.

The following tentative suggestions are made as the result of the tests:

(1) The length of the stiffener along the beam should be equal to about three-fourths the width of the flange for **H**-sections and to the full flange width for **I**-beams;

(2) The thickness of the stiffener plate material should be greater than that of the web thickness or greater than one-tenth the length of the stiffener plate along the beam;

(3) The stiffeners should be machined to a tight fit between the flanges and should be welded to flange and end plate continuously on the outer part;

(4) The end plates should have a thickness equal to twice the maximum thickness of the beam material; and

(5) The beam should be cut square and welded to the end plate with a continuous fillet weld about the entire beam end.

The stiffener plate and the weld between it and the flange should be designed to resist the shear as computed by Equation (52).

Design Examples

General Remarks.—The most economic structural **H**-beam or **I**-beam for torsional strength is one in which the material is most nearly of constant thickness throughout and is as thick and compact as obtainable. Column sections with parallel sided flanges and of the heaviest rolling in each series most nearly satisfy these requirements.

The torsional design should be made with ends assumed to be free in the case of riveted or bolted end connections; any percentage of end fixity incidentally present will simply provide an additional factor of safety. Only shearing stresses as computed by Equations (25) and (27) need be considered in free-end design.

Beams with boxed in and continuously welded end connections will be somewhat stiffer and stronger depending on the length. Both longitudinal stresses and shearing stresses must be considered (see Fig. 14). Tests indicate that the shearing stresses generally determine the yield point of the beam as a whole. The local direct stresses at the ends affect initially only a small part of the beam and are in the nature of secondary stresses. Shearing yield on the other hand occurs along the entire beam length. The allowable direct fiber stress will be made 22 000 lb per sq in. in the present discussion. It is suggested that allowable fiber stresses usual in secondary stress design be applied in general to these stresses.

General Data.—The following data apply to all the examples: Allowable working normal stress, $\sigma = 22\ 000$ lb per sq in., for secondary stresses due to fixed-end torsion; allowable working shear stress, $\tau = 12\,000$ lb per sq in.; $E = 29\ 000\ 000$ lb per sq in.; $G = 11\ 150\ 000$ lb per sq in.; Poisson's ratio, $\mu = 0.30$; $l =$ over-all length of a beam, including stiffeners, along which a uniform torsional moment is assumed to act; $1° = 0.01745$ radian; and 1 radian $= 57.30$ degrees. Computations in these problems were made with a 10-in. slide-rule.

Design Example A.—A long beam with torsional deflection limited: Assume a beam 20 ft long designed to resist a total torsional moment of 20 000 in-lb with maximum total twist under the load limited to 1.2 degrees. The procedure for designing the beam as free ended involves three steps: (1) Determine the unit angle of twist, θ, in radians per inch, thus, $\theta = \frac{1.2 \times 0.01745}{20 \times 12} = 0.0000872$ radian per inch; (2) calculate the required K-value from Equation (5), thus, $K = \frac{T}{G\theta} = \frac{20\,000}{11\,150\,000 \times 0.0000872} = 20.6$ in.^{4}; and (3) refer to standard tables of K-values[6] and pick out the most economical section. In this manner, a Bethlehem section, 10 by 10 in., at 124 lb per ft (with $K = 20.37$) will be satisfactory. The end connection will provide additional rigidity and will allow a small tolerance in picking sections.

Design Example B.—Analysis of the torsional strength of a short beam (B8b, 8 by 8, at 67 lb per ft), with different end connections: The general data applying to this case are: $l = 66$ in., over-all; $K = 5.145$ in.^{4}; $I_y = 88.6$ in.^{4}; $n = 0.933$ in.; $b = 8.287$ in.; $D = 1.206$ in.; $h = 9.000 - 0.933 = 8.067$ in.; $a = 0.806\,h\sqrt{\frac{I_y}{K}} = 27.0$ in.; $\frac{l}{2a} = \frac{66}{54} = 1.22$; $\cosh \frac{l}{2a} = 1.8412$; and $\tanh \frac{l}{2a} = 0.8397$.

The free-ended working strength is computed by Equations (25) thus:

$$T = \frac{2K\tau}{D+n}\,\frac{(2)\,(5.145)\,(12\,000)}{1.206 + 0.933} = 57\,700 \text{ in-lb.}$$

To determine the fixed-ended working strength, based on shear, compute the equivalent torsion constant, C_c, for the center of the beam by Equation (46), thus:

$$C_c = 5.145\left(\frac{1.8412}{1.8412 - 1}\right)\left(\frac{1}{1 + 2.95\left(\frac{8.29}{66}\right)^2}\right) = 10.77 \text{ in}^4.$$

Then, from Equation (47),

$$T = \frac{12\,000}{\frac{8.29^2}{4 \times 8.067 \times 88.6 \times 1.841} + \frac{1.206 + 0.933}{2 \times 10.86}} = 108\,000 \text{ in-lb}$$

[6] See, for example, Bethlehem Manual of Steel Construction, Catalog S-47, 1934, p. 285.

To determine the fixed-ended working strength, based on longitudinal fiber stresses at ends (tension or compression), apply Equation (40):

$$T = \frac{22\,000}{\dfrac{27.0 \times 8.287 \times 0.8397}{8.067 \times 88.6}} = 84\,000 \text{ in-lb}$$

The longitudinal stresses, therefore, determine the design of the beam, and the allowable torsional moment is 84 000 in-lb.

The shear to be resisted by the end plate is computed by Equation (56); thus, if s is assumed as 6.5 in.,

$$Q = \frac{84\,000 \times 27.0}{8.067 \times 6.5} \times 0.8397 = 36\,300 \text{ lb.}$$

A $\frac{5}{8}$-in. plate fitted into the 7.13-in. space between the flanges and 6 in. in length along the beam, will satisfy the requirements suggested. Assume a $\frac{5}{8}$-in. fillet weld between the stiffener and the flange. If $s = 8.827 - 3 \times \frac{5}{8} = 6.41$ in.,

$$Q = \frac{6.5}{6.41} \times 36\,300 = 36\,800 \text{ lb}$$

The stress in the plate is:

$$\tau = \frac{36\,800}{\frac{5}{8} \times 6} = 9\,800 \text{ lb per sq in.}$$

and in the weld:

$$\tau = \frac{36\,800}{\frac{5}{8} \times 0.707 \times 6} = 13\,900 \text{ lb per sq in.}$$

which is too high. Make the plate $7\frac{1}{2}$ in. long rather than 6 in. Then the stress in the weld is:

$$\tau = \frac{6}{7.5} \times 139\,000 = 11\,100 \text{ lb per sq in.}$$

which is less than the limit of 11 300 lb per sq in. recommended by the American Bureau of Welding.

Test Results

Soap Bubble Tests: Purpose and Program

The purpose of the series of tests reported herein was to evaluate, accurately, the torsion constant of structural **H**-beam and **I**-beam sections. Specifically, this problem narrowed down to determining the added torsional rigidity introduced by the juncture of two rectangles, with fillets at the re-entry corners, in excess of the torsional rigidity of these rectangles treated as separate members. The problem, therefore, was to determine α in Equation

(16). In order to establish the value of α for any shape of section it was necessary to consider two variables, $\frac{w}{n}$, the ratio of web to flange thickness, and $\frac{r}{n}$, the ratio of fillet radius to flange thickness. Furthermore, it was essential to study sections with sloping flanges as well as those with parallel sides.

A program of tests was outlined to cover a wide range of the two variables, $\frac{w}{n}$ and $\frac{r}{n}$. Although it would have been desirable to measure the slopes of the bubbles and thereby study the stresses, particularly in the fillets, such a study would have greatly reduced the total number of tests possible. It was thought better to establish the torsion constant definitely, in which case it was only necessary to measure the volume of the displaced bubble. In each series a basic web and flange thickness was adopted and after testing the section with zero fillet radius, the various fillets were cut away in sequence as shown in Fig. 16.

The curves obtained from the test results are shown in Fig. 7. It was found that scattered points occurred along the lines of 0.0 and 0.2 fillet radius. These variations may be due to the difficulty in machining the plates with the small fillets, a slight inaccuracy causing considerable variation in the diameter, D, of the inscribed circle. A further difficulty is encountered in the plates of zero fillet radius due to the tendency of the soap film to jump across these sharp corners. Most of the structural beams actually rolled have ratios of $\frac{w}{n}$ and $\frac{r}{n}$ both greater than 0.5 and in this area the data were quite consistent.

Tests of Steel Beams: Purpose and Apparatus

Torsion tests were made on steel beams, with several objects in view. The sections themselves were chosen so as to give a range of shapes and sizes as great as possible, and tests of certain unusual shapes, which are not at present standard, were nevertheless valuable in the investigation.

Tests of free-ended conditions were made on a number of beams in order to check the results of the soap film experiments and the corresponding method of calculating the torsion constant. In these tests the distribution of shearing stress was studied, and a check was obtained on the proposed approximate formulas for stress. Tests of fixed-end conditions were made on different shaped beams and the effect of variations in length was studied. A type of end-connection design was developed to give a considerable degree of fixity.

A standard torsion machine of 26 000 in-lb capacity was used to test 3-in., fixed-end I-beams, of various lengths. It was also used for torsion tests of round bar samples of all material to determine the shearing modulus of elasticity.

The cable torsion rig, shown in Fig. 17, with an ultimate capacity of 750 000 in-lb was used in the major tests.[7] Most of the large beams were tested in lengths of 6 ft, but two tests each were made on beams 1 ft 3 in. and 3 ft long by means of the same sheaves and cables adapted for use with

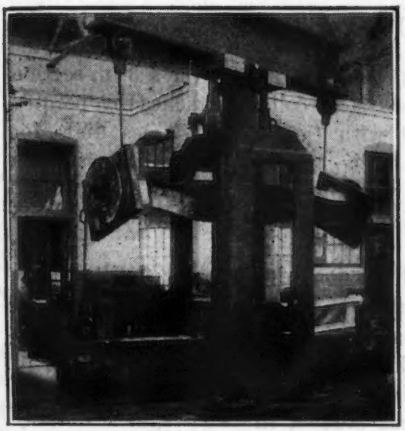

FIG. 17.—THE CABLE TORSION RING.

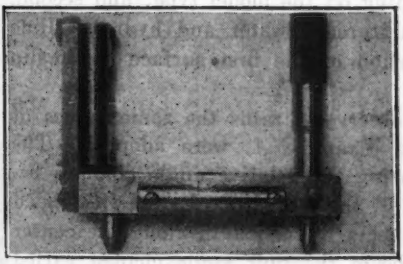

FIG. 18.—LEVEL BAR.

FIG. 19.—CONNECTION FOR FREE-ENDED BEAMS.

shorter top and bottom beams. During the tests of light beams, cables $\frac{5}{8}$ in. in diameter were used because of their flexibility and ease of handling, but in the tests of the heavier and shorter sections the cable was changed to 1 in. in diameter in order to develop the full capacity of the machine.

The sheaves were made of material 2 in. thick and were machined to a minimum diameter of 17 in. A hole bored through one of the diameters allowed continuous action and reversing of the cable without fouling or introducing bending moment. This machine gave perfect satisfaction in every respect and was easily set up and dismantled. During tests the apparatus was in such a state of balance that the heavy pulling beams could be easily tilted either way by hand while maintaining a heavy torsional load of the test specimen.

[7] "Torsion Testing Machine of 750 000-Inch-Pounds Capacity," by Bruce G. Johnston, Jun., Am. Soc. C. E., *Engineering News-Record*, February 28, 1935, p. 10.

Measuring Devices.—The level bar illustrated in Fig. 18 was built to measure the change in relative altitude of two points 3 in. apart. It was used to measure relative angle changes in all the beam tests. The micrometer vernier permitted readings to 0.0001 in. and the level bubble was sensitive to micrometer changes of about 0.0003 in. The total range of the instrument was about 9° 30′ ± from the orginal level position.

The torsion meter was used in the torsion tests of round bars in the 26 000-in-lb torsion machine to measure the angular twist over a 3-in. section of the bar. This device consisted of two steel collars attached by pointed thumb screws to the bar and provided with two 1:1 000 Ames dials for measuring the tangential displacements. The instrument was the same as that used and described in a previous investigation at the Fritz Laboratory.[8] Twenty tensometers were used to obtain the strains at all critical points during tests.

Test Procedure and Method.—In each torsion test, whether fixed or free-ended, there were three principal objectives: First, to learn as much as possible about the strain and stress distribution; second, to measure the torsional stiffness, or ratio of torsional load to angular twist; and third, to learn the useful torsional load-carrying limit of the beam as a whole.

The strain and stress distribution was studied in two ways: First, by tensometers which were sensitive to changes in strain corresponding to from 150 to 300 lb per sq in. in stress, depending on the model type; and second, the beams were whitewashed with a mixture of water and hydrated lime which showed the distribution and location of the first surface strain-slip lines.

In computing the stresses from the observed strains the same values of E, G, and μ, as those used in Design Example A, were adopted. The torsional stiffness was gauged by measuring the relative angle changes between two points along the beam by use of the level bar. In the free-ended tests the angle changes were measured over a 36-in. length along the center part of the beam. In the fixed-ended beam the unit angle change varied along the length and a measure of the average stiffness was obtained by measuring the relative rotation of the end plates and of points a short distance from each end where the reinforcement ended.

The yield point of the beam as a whole was obtained by a study of the torque-twist diagrams and a knowledge of the load when the first strain-slip lines appeared. In a few cases a definite drop of the beam was noted and, in such cases, this was taken as the yield point. In most of the tests the yield point was taken as the torsional moment corresponding to the point on the torque-twist diagram where the co-tangent of the slope was 1.5 times the value of the co-tangent of the slope of the straight part.

Test Results.—Table 1, Appendix III, gives a general summary of all the tests made, including dimensions of beams and computed K-values based on actual dimensions. The dimensions were obtained by means of micrometers

[8] "Shearing Properties and Poisson's Ratio of Structural and Alloy Steels," by Inge Lyse, M. Am. Soc. C. E., and H. J. Godfrey, *Proceedings*, A.S.T.M., 1933, Vol. 33, Pt. II.

and calipers. Readings were taken at a number of different places on the beam, and averages from them were used to calculate the weight of the beam per foot of length. Furthermore, the beams were actually weighed and any discrepancy between computed and actual weight was taken care of by adjusting the average measured dimension to give the actual weight.

Table 2, Appendix III, presents the physical properties based on tests of samples taken from the test beams. The tensile values are based on the average of two tests of American Society of Testing Materials standard tension test specimens (2-in. gauge length). The torsion tests and slotted-plate shear tests were made in the same manner as described in a previous investigation at the Fritz Engineering Laboratory.[8] In most cases, the tensile specimens and round-bar torsion specimens were cut from the material where the flange and the web join, as it is at this point that critical torsional stresses develop. The material for the slotted-plate test specimens was cut from the webs of the beams.

Free-Ended Tests.—Free-ended tests were made on seven beams. Each specimen was held in the torsion rig by two bolts at each end which passed with a loose fit through the web and through the two angles, the angles being welded to end plates which, in turn, were bolted to the sheave plates of the testing rig. Torsional moment was applied by means of lugs welded to the end plates which engaged the flanges of the test specimens. The flanges

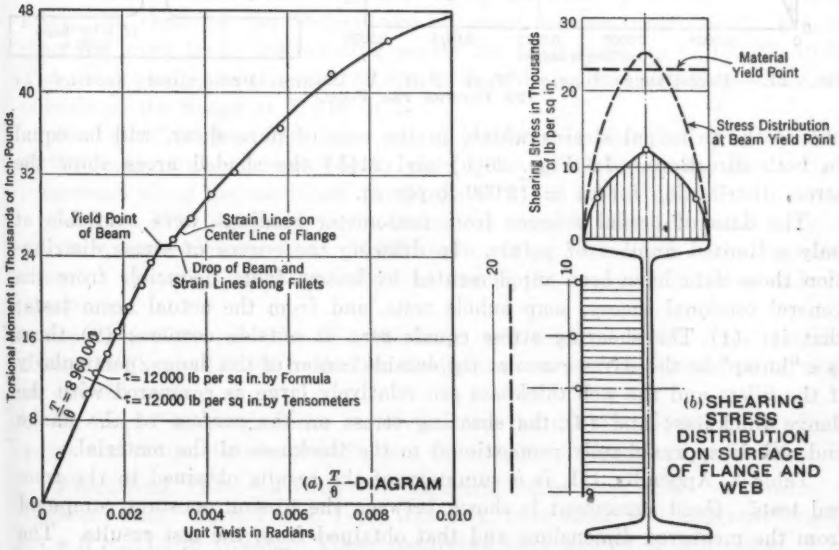
Torsional Moment in Thousands of Inch-Pounds
Unit Twist in Radians
Yield Point of Beam
Strain Lines on Center Line of Flange
Drop of Beam and Strain Lines along Fillets
τ = 12000 lb per sq in. by Formula
τ = 12000 lb per sq in. by Tensometers
(a) T/θ - DIAGRAM
Shearing Stress in Thousands of lb per sq in.
Material Yield Point
Stress Distribution at Beam Yield Point
(b) SHEARING STRESS DISTRIBUTION ON SURFACE OF FLANGE AND WEB

FIG. 20.—FREE-ENDED TORSION TEST *T*-26, A 12-INCH I-BEAM.

were thus free to warp and the beams were almost entirely unrestrained at the ends. Fig. 19 shows the details at one end of the largest beam tested and is typical of all the free-end tests which were made. Figs. 20(*a*) and 21(*a*) show typical torque-twist diagrams of two of the tests. Figs. 20(*b*)

and 21(*b*) show the stress distribution based on tensometer readings taken in these same tests. The tensometers were placed on the flange and web surface at about the center cross-section of the beams and were set at an angle of 45° with the longitudinal axis of the beam, in order to measure

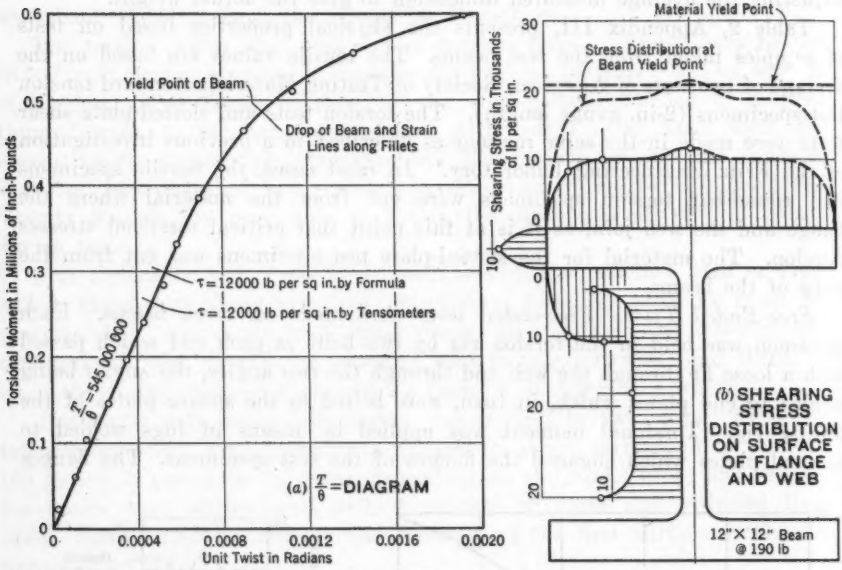
Torsional Moment in Millions of Inch-Pounds
Unit Twist in Radians
Yield Point of Beam
Drop of Beam and Strain Lines along Fillets
τ = 12000 lb per sq in. by Formula
τ = 12000 lb per sq in. by Tensometers
T/θ = 545 000 000
(a) T/θ = DIAGRAM
Material Yield Point
Stress Distribution at Beam Yield Point
Shearing Stress in Thousands of lb per sq in.
(b) SHEARING STRESS DISTRIBUTION ON SURFACE OF FLANGE AND WEB
12" × 12" Beam @ 190 lb

FIG. 21. — FREE-ENDED TORSION TEST *T*-31, A 12-INCH BETHLEHEM SECTION AT 190 POUNDS PER FOOT.

one of the principal strains which, in the case of pure shear, will be equal in both directions. In Figs. 20(*b*) and 21(*b*) the shaded areas show the stress distribution for $\tau_m = 12\,000$ lb per sq. in.

The data of actual stresses from tensometer readings were available at only a limited number of points. In drawing the curves of stress distribution these data have been supplemented by known facts, deducible from the general torsional theory, soap bubble tests, and from the actual beam tests; that is: (1) The shearing stress equals zero at outside corners; (2) there is a "hump" in the stress curve at the outside center of the flange, particularly if the fillets and the web thickness are relatively large as compared with the flange thickness; and (3) the shearing stress on the surface of the flange and web is approximately proportional to the thickness of the material.

Table 3, Appendix III, is a summary of the results obtained in the free-end tests. Good agreement is shown between the torsion constant computed from the measured dimensions and that obtained from the test results. The test value for K was obtained from the slope of the torque-twist diagram and from Equation (5); thus, $T = K G \theta$; or, $K = \frac{T}{G\theta}$.

The maximum variation for the K-value of test results was 6.7 and the average variation of seven tests was 2.26 per cent. It is noted that

K for the heaviest beam tested was about two hundred times greater than for the lightest beam and that the corresponding agreement for these tests was expressed by a variation of 0.9 and 0.0 per cent.

The shearing stress computed by Equation (25) gave average stresses 7% less than those based on tensometer readings. The stresses in the web by Equation (26) which, theoretically, should be correct, were much lower than as computed from tensometer readings. Equation (27) was suggested on the basis of these tests which, of course, are not complete enough to substantiate its adoption definitely. However, both Equations (25) and (27) are believed to be usable for practical design purposes with ordinary values of allowable shearing unit stress. The following special remarks apply to the individual free-end tests:

In Test T-12 strain lines appeared along the fillets at 13 500 in-lb; and along the outside center line of the flange at 15 200 in-lb. The yield point of the beam as determined by the slope of the torque-twist diagram was 15 900 in-lb.

In Test T-22 the freedom of the ends from restraint was checked by tensometers placed longitudinally near the ends. The strains were negligible. The first shear strain line along the center line of the flange appeared at 21 210 in-lb. At 25 610 in-lb, strain lines progressed rapidly along the flange and in the fillets, and a definite drop of the beam was noted.

In Test T-25 strain lines appeared along the fillets at 50 900 in-lb. Thereafter, the slope of the torque-twist diagram became nearly 50% greater than for lower loads, maintaining nearly the same slope up to 100 000 in-lb. The yield point was taken as 50 900 in-lb. Strain lines appeared along the outside of the flange at 77 340 in-lb.

In Test T-26 (see Fig. 20) a slight checking in the fillets was noted at 23 000 in-lb, with a drop-of-beam yield point at 25 000 in-lb; strain lines progressed along the outside of the flange at 26 500 in-lb.

In Test T-30 the yield of the beam was noted by the 1.5 on 1.0 slope of the torque-twist diagram at 64 000 in-lb. The first strain lines appeared over the web at an indeterminate load due to the presence of heavy scale on the section.

In Test T-31 (see Fig. 21) the beam and dropped strain lines occurred along the fillets at 48 800 in-lb. The yield point was noted by the 1.5 on 1.0 slope at 480 000 in-lb.

In Test T-33 strain lines occurred along the fillets at 65 600 in-lb. Yield point and strain lines appeared along the outside of the flange at 76 400 in-lb.

Fixed-End Tests.—Twenty-two fixed-end tests were made on beams of different sizes. Tests T-4 to T-13, inclusive, were made with different lengths of a 3-in., 7.5-lb I-beam. These specimens were welded to end plates, 1 in. thick, with a continuous $\frac{3}{8}$-in. fillet weld. The end plates were bolted to $1\frac{1}{4}$-in. plates which were attached to the jaws of the standard 26 000 in-lb torsion machine. The relative rotation of the end plates was measured by level-bar readings taken at each end. The remainder of the beams were tested in the cable-testing rig. All these beams were larger than the 3-in. I-beams

(a) $\frac{T}{\theta}$-DIAGRAM
Torsional Moment in Thousands of Inch-Pounds
Unit Twist in Radians
Yield Point of Beam
$\frac{T}{\theta}$ = 33 700 000
(b) SHEARING STRESS AT SECTION a-a AT THE YIELD POINT STRESS IN THE BEAM
(c) TOP VIEW OF FLANGE
Center Line
Shear Tensometers
Tensometers for Direct Stress
Stiffeners
6'0"
9"
5"
(d) DIRECT STRESS IN OUTER FIBERS OF FLANGE AT YIELD POINT OF THE BEAM
Compression in Top Outer Fiber
Tension in Top Outer Fiber
Normal Stress in Thousands of lb per sq in.

FIG. 22.—FIXED-ENDED TORSION TEST *T*-17, A 6-INCH H-BEAM AT 41 POUNDS PER FOOT.

and were welded to $1\frac{1}{4}$-in. end plates with additional end stiffeners, fitted and welded between the flanges, as was illustrated in Fig. 15.

Relative rotation of the end plates was observed on all beams by means of level-bar observations, and the twist of the large beams having stiffener plates was also measured at points just inside those plates. Strain readings for longitudinal and shearing strains were observed wherever feasible. Fig. 22(*a*) shows a typical fixed-end torque-twist diagram, and Fig. 22(*b*), 22(*c*), and 22(*d*) show the computed stresses from strain readings at the yield point of the beam. These observations are typical of all the fixed-end tests.

Table 4, Appendix III, gives the summary of the test results for fixed-end beams. In computing the values of C_c and C_A the question arose as to the correct length to be used. If 100% end fixity were possible the correct length would be slightly less than the over-all length and somewhat greater than the length between the end stiffener plates. However, the over-all length is the simplest approximation, and it gives the best results by comparison with the tests, except in the case of very short beams with end stiffeners. In these two tests (*T*-19 and *T*-27, Table 4) the apparent percentage of end-fixity seems inconsistently high. Two tests have unusually low percentages of end fixity (see Tests *T*-16 and *T*-24). The explanation for this is given under the special remarks. The average percentage of end fixity with Tests *T*-16 and *T*-24 omitted is 88.3 and all the 6-ft beams, except *T*-24, have an end efficiency greater than 85 per cent.

In most cases the yield points of the beams were determined from the slope of the torque-twist diagrams and the theoretical direct stresses computel on the basis of this yield-point torque are given in Column (14), Table 4 (Appendix III). It is noted that, in spite of incomplete end fixity, these stresses, in every case, are above the tensile yield-point strength of the material as given in Table 2 (Appendix III). Hence, all the beams

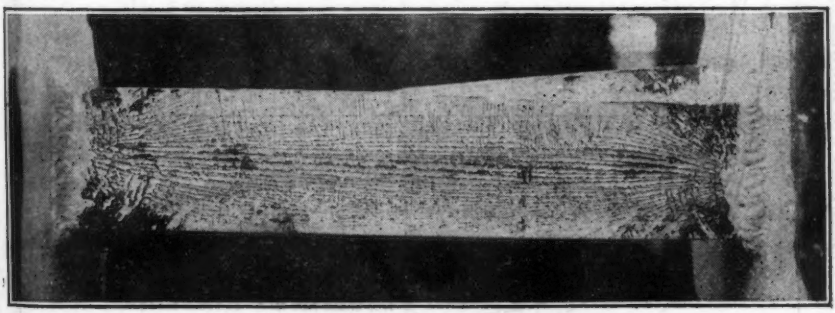

FIG. 23.—ILLUSTRATION OF STRAIN PATTERN.

would have been designed safely on the basis of working, direct, fiber stresses. The average of Column (14), Table 3 (Appendix III), is 55% more than the average yield-point strength of the material in the test beams.

The computed and measured shearing stresses in the flange agree well for all the 6-ft beams, with the exception of *T*-16 and *T*-24 in which the low

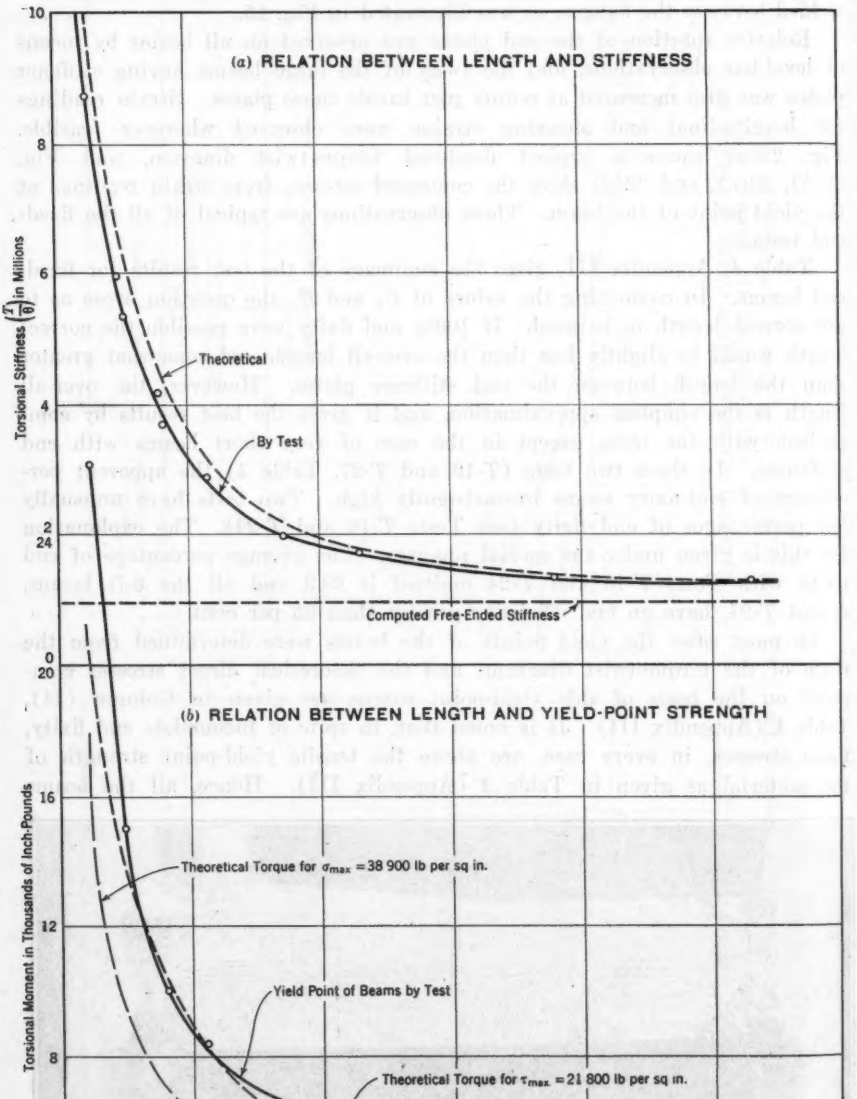

FIG. 24.—TEST OF 3-INCH I-BEAM AT 7.5 POUNDS PER FOOT, FIXED AT BOTH ENDS.

end fixity affects, directly, the shearing stress agreement. On the short 18-in. beams with stiffener plates (*T*-19 and *T*-27), the discrepancy between the computations and the test results is high, as might be expected. The computed stresses in the web give an approximate check on the stresses indicated by tensometers. Special remarks on fixed-end tests are, as follows:

Tests *T*-4 to *T*-12 were of different lengths of the same 3-in. I-beams and were well adapted to show the influence of end fixity on the strength and stiffness with the length of beam the only variable. The flange of each beam was whitewashed so that the appearance of first strain lines might be noted. Fig. 23 shows the strain line pattern on the flange of one of these beams after yield had taken place. Fig. 24(*a*) gives a graph of test results for this series showing the influence of length upon the rigidity, and Fig. 24(*b*) illustrates the influence of length upon the yield-point strength of the beams. Special attention in this series of tests was given to Tests *T*-8 and *T*-10. Stress measurements were taken along the extreme fiber of the flanges at short intervals of length and the lateral bending moment in the flanges for a definite torque load was computed from these readings. The bending moments along the beam were plotted and the curves were differentiated to give the lateral shear in the flanges. These results are compared in Fig. 25 with the theoretical variation in shear by Equation (38).

Tests *T*-15 and *T*-16 should be compared with the free-end test, *T*-14, of the same section. Test *T*-16 was a special run with additional stiffeners placed midway along the section. These stiffeners were of the same type as the end stiffeners and were parallel in a plane with the web. No outstanding stiffener was provided. Although this additional stiffener gave the beam an average stiffness 42% greater than Test *T*-15, it provided only 40.1% of the theoretical stiffness of a beam 3 ft in length, rigidly fixed at each end. The design of this beam would have been safe for strength, however, if based on the 3-ft length and designed for the proper longitudinal working stress.

Test *T*-17, in contrast to Test *T*-15, was of the heaviest 6-in. section rather than the lightest. It should be noted that an effective fixity of 96.8% was attained in this test. The end stiffeners were $\frac{5}{8}$ in. thick and 5 in. long. The torque-twist diagram and data on stress distribution are given in Fig. 22 as typical of the results for fixed-end tests.

Test *T*-18, of a subway column, provided an opportunity to observe a section of extreme proportions.

Tests *T*-19 to *T*-21, together with free-end Test *T*-22 on the same sized section, provided a series of different lengths of 8-in. H-section. It is noted that the shortest beam tested showed an end-fixity efficiency of 101%, whereas the general trend for shorter beams should be less end fixity because of the greater strains placed upon the end connection. This effect is explained by the fact that the over-all length was used in the computations. Although this is good approximation for the longer beams, the stiffness changes rapidly in the short length range and the correct length is some

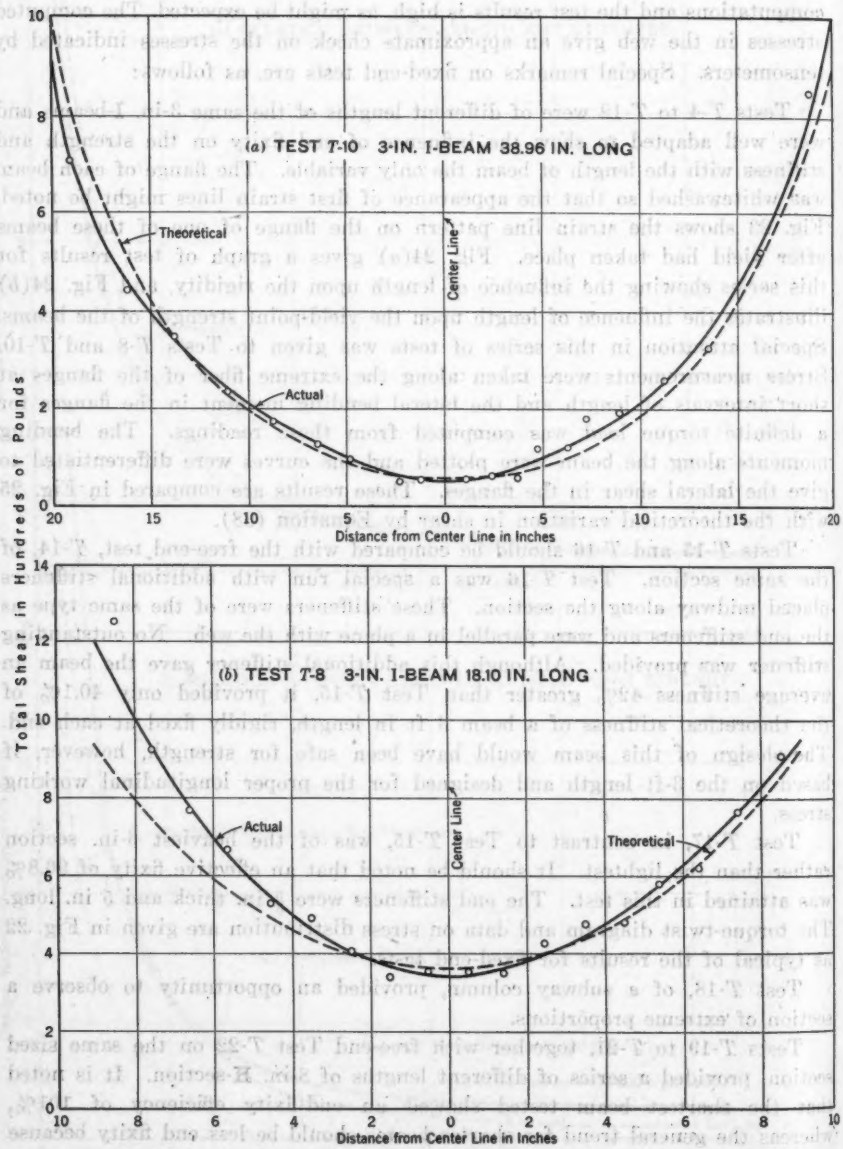

FIG. 25.—SHEAR DISTRIBUTIONS IN FLANGES OF FIXED-ENDED BEAMS FOR A TORSIONAL MOMENT OF 2 500 INCH-POUNDS.

value less than that used. Fig. 26 shows the strain-line distribution near the end of Beam *T*-21 after yielding had occurred.

Test *T*-23 is of the heaviest section of the 8-in. H-sections, whereas Tests *T*-19 to *T*-22 were of the lightest weight section.

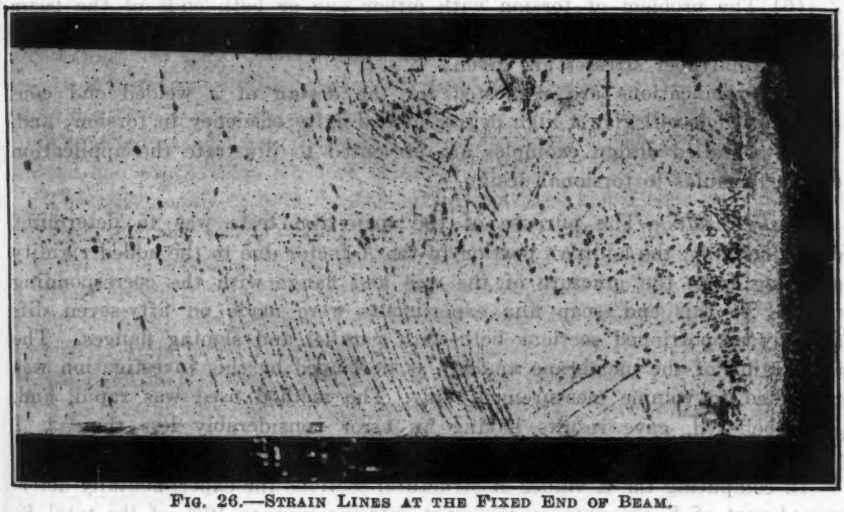

FIG. 26.—STRAIN LINES AT THE FIXED END OF BEAM.

Test *T*-24 is of interest because it had the lowest end-fixity efficiency of the 6-ft beam sections, with an end fixity of only 45.5 per cent. The remaining 6-ft sections were more than 85% efficient in end fixity. The explanation is found in the fact that the end stiffeners in this base were not properly designed, and the beam not only pulled loose at the weld from the end plate, but the end plate was badly warped during the test. The stiffeners were $\frac{5}{16}$ in. thick and 6 in. long. Had they been 9 in. long and 1 in. thick, as provided in the tentative Design Suggestions (1) and (2) (see heading, "Design of End Connection"), it is believed that much higher efficiency would have resulted.

Tests *T*-27 to *T*-29 are of three different lengths of 12-in. I-beam. The remarks concerning Tests *T*-19 to *T*-21 appear to apply in the case of these beams also.

SUMMARY AND CONCLUSIONS

Torsional Theory and the Torsion Constant.—The material presented in this paper may be outlined under eight divisions, as follows:

(1) The essential features of the general torsion problem are outlined;

(2) The application of Prandtl's membrane analogy is presented;

(3) An accurate and detailed method of evaluating the torsion constant of structural H-beams and I-beams is presented (this method is based on a known theoretical evaluation combined with factors determined by experiment from the membrane analogy);

(4) Formulas are proposed for the shearing stress in the flange and web due to pure torsion;

(5) The effect of shearing stress concentration in the fillets due to both torsion and bending is discussed;

(6) The problem of torsion with either one or both ends of the beam restrained is studied in detail and formulas for maximum shearing and longitudinal fiber stresses are given;

(7) Specifications are suggested for the design of a welded end connection which will give a high degree of end-fixity efficiency in torsion; and,

(8) Detailed design examples are presented to illustrate the application of the formulas to torsional design.

Test Results.—The purpose of the soap film tests was to determine, experimentally, the additive portion in the K-factor due to the added rigidity accruing from the juncture of the web and flange with the corresponding fillets. To this end, soap film experiments were made on fifty-seven differently proportioned sections both with parallel and sloping flanges. The application of the membrane analogy as developed by this investigation was restricted to volume measurement only. The method used was rapid, and, it is believed, gave results having an error considerably less than $\pm$ 1 per cent.

In computing the K-values of structural sections the experimentally determined part of K amounts, in the most extreme case, to 10% of the total K. Hence, an experimental error of $\pm$ 1% would give a possible error of only 0.1% in the total K-value.

Free-end torsion tests were made on seven different beams ranging in size from a 6-in. **H**-beam @ 20 lb per ft to a 12-in. **H**-beam @ 190 lb per ft. The heaviest beam had a torsion constant about two hundred times as great as the lightest beam.

The method of applying the torque to the ends of the beam provided a high degree of end freedom. By measuring the unit twist of the free-ended beams and obtaining the slope of the torque-twist diagram, the free-ended torsion tests provided (through Equation (5)) a definite check on the torsion constant as computed by the proposed method. Using $G = 11\,150\,000$ lb per sq in., which is theoretically correct for $E = 29\,000\,000$ lb per sq in. and $\mu = 0.30$, the test results checked well with computed K-values. The maximum variation was 6.7% and the average for the seven tests was 2.26 per cent.

The yield point of the beams in both free-ended and fixed-ended tests was determined by a study of the torque-twist diagram and in some cases by a drop of the beam. In the free-ended tests the distribution of shearing stress across the flange was studied by measuring, with tensometers, the principal stress at an angle of 45° around the center section.

The fixed-ended tests provided a means of studying the additional rigidity and strength over the free-ended tests, due to fixing the ends of the beams. Twenty-two various lengths and sizes of beams were tested ranging from

a 3-in. I-beam at 7.5 lb per ft, to a 10 by 12-in. beam at 62 lb per ft. The heaviest beam had a K-factor fifty-seven times greater than the least, and with ends fixed the most rigid fixed-end beam had a rigidity three hundred and ten times as great as the least rigid with ends fixed.

The distribution of longitudinal direct stresses in the extreme fibers of the flanges was studied by strain measurements taken along the outer edges of the flanges by tensometers. The total shearing stress distribution at the center section was studied by measurement of principal strains at an angle of 45 degrees.

The fixed-ended tests furnished information as to the proper design of welded end connections for high torsional rigidity.

The tensometers in both the free-ended and fixed-ended tests were of value in studying the relative distribution of stress. Some of the results are erratic. The warping of the section during twist made it difficult to obtain a steady set-up for the tensometers, but in most cases the test results checked fairly well with the formulas. In spite of variations in result and incomplete end fixity it is noted that every beam tested would have been amply strong if designed on a basis of working longitudinal stresses at the ends.

The present investigation has covered, accurately, the question of torsional rigidity and the evaluation of the torsion constant. The formulas for stresses which are proposed are not exact but will be satisfactory for practical design purposes.

Acknowledgment

The torsional investigation was conducted as a co-operative project, the McClintic-Marshall Corporation, subsidiary to the Bethlehem Steel Company, furnishing the steel beams and the material for constructing the test apparatus, and Lehigh University bearing all other expenses.

Through the courtesy of the Baldwin-Southwark Corporation, of Philadelphia, Pa., sixteen tensometers were loaned to the University (without charge), during these tests.

Special acknowledgment is due Jonathan Jones, M. Am. Soc. C. E., Chief Engineer; C. H. Mercer, M. Am. Soc. C. E., Consulting Engineer, and Sterling Johnston, M. Am. Soc. C. E., Engineer, all of the McClintic-Marshall Corporation, and V. E. Ellstrom, Manager of Sales Engineering, Bethlehem Steel Company, for their co-operation in the investigation.

The study was conducted as a regular research project of the Fritz Engineering Laboratory, Lehigh University. C. C. Keyser, Laboratory Assistant, and all others associated with the Laboratory contributed valuable help throughout. Professor J. B. Reynolds, of the Department of Mathematics, gave his assistance to the theoretical studies.

APPENDIX I

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APPENDIX II

NOTATION

The following symbols, adopted for use in this paper, are presented as a guide to discussers:

a = a torsional factor in fixed-ended beams.
b = length of a rectangular section.
d = depth; total depth of beam.
e = eccentricity.
f = a subscript denoting "flange."
h = distance between flange centroids.
$i = \frac{1}{2}(d - h)$.
m = major flange thickness.
n = breadth of a rectangular section; also, where defined, n = minor flange thickness.
p = pressure per unit area.
r = radius of a fillet (authors' duplicate); also, where defined, r = variable thickness of a section.
s = distance between stiffening plates.
t = thickness; as a subscript, t, denotes "due to bending."
u = a substitution factor in Equation (37); as a subscript, u, denotes "free ended."
w = web thickness; as a subscript, w, denotes "web."
y = deflection.
A = area; cross-section area; also, where specifically defined, A, B, C, and D are coefficients in a general equation; as a subscript, A, denotes "average."
B = a substitution factor in Equation (23) (see, also, Symbol A).
C = constant; torsion constant equivalent to K for a fixed-ended beam; $C_c = C$ at the center; C_A = average value of C (see, also, Symbol A).
D = diameter; diameter of an inscribed circle (see, also, Symbol A).
E = elasticity; modulus of elasticity in tension and compression.
F = torsion stress function.
G = shearing modulus of elasticity.
I = rectangular moment of inertia; I_x and $I_y = I$ for a cross-section with respect to the X-axis and the Y-axis, respectively.
J = polar moment of inertia.
K = a torsion constant; K_f and K_w = values of K for flange and web, respectively.
L = length; as a subscript, L, denotes "large end."
M = bending moment.
Q = total shear over a cross-section.
S = total slope of a flange section $= \frac{m - n}{b}$; as a subscript, S, denotes "small end."
T = torque; torsional moment; T_c = torque required to twist a beam in a free-ended condition.

V = a factor depending on the $\frac{b}{n}$ -ratio, but practically constant for $\frac{b}{n}$ greater than 3; V_L and V_S = value of V for large end and small end, respectively, of the flange.

α = a factor that depends on two ratios, $\frac{w}{m}$ and $\frac{r}{m}$.

ϵ = unit elongation.

θ = angle of twist, in radians per unit length; $\theta_c = \theta$ at the center.

μ = Poisson's ratio.

ρ = radius of curvature.

σ = unit stress; normal stress; σ_t = longitudinal fiber stress in a flange due to bending.

τ = unit shearing stress; $\tau_f = \tau$ for flange sections; $\tau_w = \tau$ for web sections.

ψ = total angle change between two cross-sections.

APPENDIX III

TABLES

The tables presented herewith are introduced and adequately discussed in the text of this paper.

TABLE 1.—GENERAL SUMMARY OF ALL TESTS

Test No.	Nominal size	Weight, in Pounds per Foot		Length, in inches	Measured Dimensions, in Inches, Adjusted to the Actual Weight (See Fig. 6)						Measured diameter, D, of inscribed circle, in inches	Torsion constant, K
		Nominal	Actual		Total depth, d	Length, b	Web thickness, w	Flange thickness				
								m	n			
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
T-4*	3-in. I-beam	7.5	7.4	3.2	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-5*	3-in. I-beam	7.5	7.4	6.0	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-6*	3-in. I-beam	7.5	7.4	9.0	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-7*	3-in. I-beam	7.5	7.4	12.1	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-8*	3-in. I-beam	7.5	7.4	18.1	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-9*	3-in. I-beam	7.5	7.4	23.9	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-10*	3-in. I-beam	7.5	7.4	39.0	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-11*	3-in. I-beam	7.5	7.4	53.9	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-12*	3-in. I-beam	7.5	7.4	5.5	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-13*	3-in. I-beam	7.5	7.4	8.8	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-14†	6 by 6-in.	20	19.6	72	6.06	6.01	0.257	.....	0.362	0.28	0.566	0.243
T-15*	6 by 6-in.	20	19.6	72	6.06	6.01	0.257	.....	0.362	0.28	0.566	0.243
T-16*	6 by 6-in.	20	20.9	72	6.08	6.06	0.303	.....	0.372	0.24	0.581	0.281
T-17*	6 by 6-in.	40.5	39.0	72	6.73	6.25	0.501	.....	0.699	0.35	0.928	1.738
T-18*	6 by 10-in.	40	38.1	72	6.20	9.91	0.431	.....	0.447	0.35	0.735	0.802
T-19*	8 by 8-in.	31	29.4	18	8.05	8.04	0.290	.....	0.401	0.41	0.676	0.463
T-20*	8 by 8-in.	31	29.4	36	8.05	8.04	0.290	.....	0.401	0.41	0.676	0.463
T-21*	8 by 8-in.	31	29.4	72	8.05	8.04	0.290	.....	0.401	0.41	0.676	0.463
T-22†	8 by 8-in.	31	29.4	72	8.05	8.04	0.290	.....	0.401	0.41	0.676	0.463
T-23*	8 by 8-in.	67	66.1	72	9.05	8.29	0.606	.....	0.907	0.42	1.208	4.912
T-24*	10 by 12-in.	62	60.9	72	10.05	11.92	0.390	0.659	0.546	0.45	0.919	2.101
T-25†	10 by 12-in.	62	60.9	72	10.05	11.92	0.390	0.659	0.546	0.45	0.919	2.101
T-26†	12-in. I-beam	31.8	31.3	72	12.06	4.97	0.350	0.705	0.345	0.42	0.882	0.833
T-27*	12-in. I-beam	55	53.6	18	12.07	5.74	0.730	0.900	0.449	0.54	1.244	3.318
T-28*	12-in. I-beam	55	53.6	36	12.07	5.74	0.730	0.900	0.449	0.54	1.244	3.318
T-29*	12-in. I-beam	55	53.6	72	12.07	5.74	0.730	0.900	0.449	0.54	1.244	3.318
T-30†	12-in. I-beam	55	53.6	72	12.07	5.74	0.730	0.900	0.449	0.54	1.244	3.318
T-31†	12 by 12-in.	190	186.7	72	14.39	12.56	1.069	.....	1.714	0.62	2.186	48.437
T-33†	12 by 12-in.	65	65.6	72	12.14	12.03	0.409	.....	0.604	0.58	0.953	2.227

* Fixed end. † Free end.

TABLE 2.—Physical Tests of Material in Test Beams

Samples; taken from Test Beam No.	Tensile Properties: Stresses, in Pounds per Square Inch: Upper yield point	Lower yield point	At ultimate	Modulus of elasticity E, in thousands of pounds per square inch	Percentage elongation in 2 inches	Percentage reduction in area	Shearing Properties, in Pounds per Square Inch: Slotted Plates: Shearing yield point	Shearing ultimate	Round Bar Torsion Tests: Shearing yield point	Apparent shearing ultimate	Shearing modulus, G, in thousands of pounds per square inch
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
T-4 } T-13 }	38 900	.....	63 550	29 100	32.0	62.1	21 800	52 850	.....	.....	.....
T-14 } T-15 }	41 000	37 860	60 450	29 200	35.8	66.4	25 600	48 500	26 400	80 600	12 080
T-16	39 710	35 790	58 130	29 150	35.5	66.7	22 000	45 100	22 000	50 700	11 280
T-17	43 020	37 000	60 220	28 500	37.0	67.3	23 700	49 900	24 000	64 100	12 020
T-18	34 180	31 570	56 570	28 250	40.0	65.4	22 600	47 900	22 200	65 600	11 500
T-19 } T-22 }	42 880	38 290	61 280	29 450	36.0	66.1	25 500	51 000	22 300	64 700	10 780
T-23	34 890	31 580	61 180	28 600	38.8	66.1	21 200	50 700	20 200	65 700	11 780
T-24 } T-25 }	40 860	35 530	59 600	29 150	36 3	64 9	23 700	49 800	24 400	64 100	12 350
T-26	38 310	33 900	56 330	29 400	38.0	69.3	24 600	52 800	21 400	71 000	11 920
T-27 } T-30 }	32 380	30 990	59 850	29 400	34.5	65.9	18 200	47 700	20 900	66 200	11 770
									18 700	64 500	10 780
T-31	32 910	28 620	60 000	29 200	38.0	63.7	21 200	51 600	20 300	64 600	11 260
Average..	38 090	34 110	59 740	29 040	36.5	65.8	22 740	49 800	22 070	65 620	11 590

TABLE 3.—Tests on Free-Ended Beams

Test No.	Nominal size	Nominal weight, in pounds per foot	Torsion Constant, K: From measured dimensions	By test	Percentage variation	Yield point of beam, in inch-pounds	Shearing Stresses, in Pounds per Square Inch: Maximum, at the Yield Point: In the Flange: By Equation (25)	By tensometers	In the Web: By Equation (27)	By tensometers	Yield point strength of material
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
T-14.....	6 by 6-in.....	20	0.243	0.243	0.0	15 900	30 400	29 700	22 400	19 700	26 000
T-22.....	8 by 8-in.....	31	0.463	0.451	−2.6	25 610	29 800	31 400	22 600	22 200	23 900
T-25.....	10 by 12-in...	62	2.101	1.960	−6.7	50 900	19 100	23 400	12 600	17 900	24 050
T-26.....	12-in I-beam..	31.8	0.833	0.804	−3.5	25 000	23 800	25 400	14 900	17 700	23 000
T-30.....	12-in. I-beam..	55	3.318	3.380	+1.9	64 000	20 700	20 200	17 100	18 500	19 000
T-31.....	12 by 12-in...	190	48.440	48.870	+0.9	480 000	19 300	21 400	12 400	12 600	20 750
T-33.....	12 by 12-in...	65	2.227	2.222	−0.2	76 400	26 600	29 300	.....	.....	.....
Average..	.....	....	.....	.....	2.26	.....	24 240	25 830	17 000	18 100	22 780

TABLE 4.—TESTS OF FIXED-ENDED BEAMS

Test No.	Nominal size	Nominal weight, in pounds per foot	Length of beam, l, in inches	Moment of inertia, I_y, in inches⁴ (from measured dimensions)	Torsion constant, K, from measured dimensions	Values of coefficient, $a = 0.806\,h\sqrt{\frac{I_y}{K}}$	Values of ratio, $\frac{l}{2a}$	Torsion Constants: C_s, computed by Equation (46)	Torsion Constants: C_A Computed	Torsion Constants: C_A By test
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
T -4	3-in. I-beam	7.5	3.2	0.57	0.0865	5.67	0.282	0.7998	0.943	0.787
T -5	3-in. I-beam	7.5	6.0	0.57	0.0865	5.67	0.529	0.4558	0.605	0.474
T -6	3-in. I-beam	7.5	9.0	0.57	0.0865	5.67	0.794	0.2823	0.394	0.330
T -7	3-in. I-beam	7.5	12.1	0.57	0.0865	5.67	1.067	0.1996	0.284	0.258
T -8	3-in. I-beam	7.5	18.1	0.57	0.0865	5.67	1.596	0.1341	0.190	0.174
T -9	3-in. I-beam	7.5	23.9	0.57	0.0865	5.67	2.108	0.1102	0.154	0.153
T-10	3-in. I-beam	7.5	39.0	0.57	0.0865	5.67	3.439	0.0913	0.120	0.116
T-11	3-in. I-beam	7.5	53.9	0.57	0.0865	5.67	4.753	0.0875	0.109	0.111
T-12	3-in. I-beam	7.5	5.5	0.57	0.0865	5.67	0.485	0.5004	0.654	0.534
T-13	3-in. I-beam	7.5	8.8	0.57	0.0865	5.67	0.776	0.2904	0.404	0.377
T-15	6 by 6-in.	20	72.0	13.1	0.2433	33.7	1.068	0.6186	0.905	0.777
T-16	6 by 6-in.	20	36.0*	13.8	0.2810	32.4	0.556	1.896	2.761	1.106
T-17	6 by 6-in.	40.5	72.0	28.5	1.7375	19.7	1.872	2.431	3.445	3.333
T-18	6 by 10-in.	40	72.0	72.5	0.8018	44.1	0.816	2.917	4.264	3.944
T-19	8 by 8-in.	31	18.0	34.8	0.4631	53.5	0.168	20.927	27.55	27.90
T-20	8 by 8-in.	31	36.0	34.8	0.4631	53.5	0.336	7.496	10.75	7.43
T-21	8 by 8-in.	31	72.0	34.8	0.4631	53.5	0.673	2.346	3.45	3.30
T-23	8 by 8-in.	67	72.0	86.2	4.9122	27.4	1.314	9.479	13.67	12.00
T-24	10 by 12-in.	62	72.0	162.7	2.101	67.0	0.537	15.104	22.03	10.02
T-27	12-in. I-beam	55	18.0	18.9	3.318	21.8	0.413	32.059	44.30	34.50
T-28	12-in. I-beam	55	36.0	18.9	3.318	21.8	0.826	11.636	16.89	11.23
T-29	12-in. I-beam	55	72.0	18.9	3.318	21.8	1.651	5.1675	7.39	6.58

Test No.	End fixity (percentages)	Yield point of beam, by test in inch-pounds	Unit Stresses, in Pounds per Square Inch: Maximum direct stress at end, by Equation (40)	Direct stress near end by tensometers	Maximum Shearing Stress at the Yield Point: In the Flange: By Equation (47)	In the Flange: By tensometers	In the Web: By Equation (48)	In the Web: By tensometers
(1)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
T -4	83.5	26 000	58 700	.....	30 500	.....	.....	.....
T -5	78.3	16 000	70 500	.....	24 500	.....	.....	.....
T -6	83.8	10 000	60 300	.....	20 360	.....	.....	.....
T -7	90.8	8 500	61 100	.....	22 000	.....	.....	.....
T -8	91.6	6 500	54 600	.....	22 800	.....	.....	.....
T -9	99.4	6 160	54 500	.....	25 400	.....	.....	.....
T-10	96.3	5 500	50 100	.....	26 400	.....	.....	.....
T-11	101.8	5 400	49 300	.....	27 000	.....	.....	.....
T-12	84.2	8 000†	32 900†	.....	11 600†	.....	.....	.....
T-13	93.4	6 000†	35 600†	.....	12 000†	.....	.....	.....
T-15	85.8	33 980	72 600	66 100†	28 000	29 600	18 700	18 500
T-16	40.1	35 000	44 100	61 500†	11 100	23 200	.....	.....
T-17	96.8	79 000	54 000	40 000	27 800	29 000	19 700	16 400
T-18	92.5	69 000	48 700	30 300	17 000	16 700	.....	.....
T-19	101.0	151 600	40 700	49 000	13 000	25 800	.....	.....
T-20	69.1	115 000	60 000	56 700	14 900	19 100	.....	.....
T-21	95.7	65 000	61 600	56 300†	18 100	19 500	.....	.....
T-23	87.8	194 000	53 300	37 400	24 000	24 800	15 000	14 500
T-24	45.5	223 000	57 000	48 100	16 200	21 600	7 800	13 450
T-27	77.9	310 000	71 000	29 400	17 700	30 000	8 600	5 920
T-28	66.6	162 000	64 300	68 500	17 900	24 400	12 400	13 800
T-29	89.0	105 000	57 100	65 800	22 700	20 200	18 100	23 740

* 36 in. = one-half total length on account of additional stiffener in middle. † Not yield point.

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DISCUSSIONS

STRUCTURAL BEAMS IN TORSION

Discussion

BY MESSRS. H. M. WESTERGAARD AND R. D. MINDLIN, AND JOSEPH B. REYNOLDS

H. M. WESTERGAARD,[12] M. AM. SOC. C. E., and R. D. MINDLIN,[13] JUN. AM. SOC. C. E. (by letter).[13a]—In Fig. 9 the authors show curves which represent different formulas for the concentration of torsional shearing stresses at fillets. The curve which is identified by the names of the writers was derived as a result of correspondence with the authors, and for this reason its derivation is given herein.

The process of approximate analysis is closely related to those used by A. and L. Föppl[14] and by S. Timoshenko,[15] but differs in the following respects:

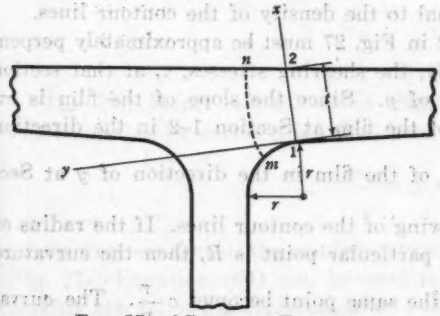

FIG. 27.—I-BEAM IN TORSION.

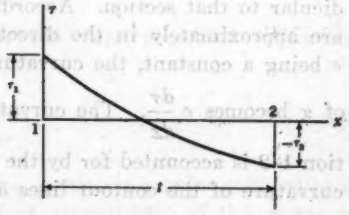

FIG. 28.—SHEARING STRESSES AT SECTION 1-2, IN FIG. 27.

First, consideration of the influence of the straight edge of the cross-section opposite the fillet led to a slightly greater concentration factor near the beginning of the fillet, at Point 1 in Fig. 27, than that obtained by Timoshenko's

NOTE.—The paper by Inge Lyse, M. Am. Soc. C. E., and Bruce G. Johnston, Jun. Am. Soc. C. E., is published in this number of *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

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[13a] Received by the Secretary March 1, 1935.

[14] "Drang und Zwang," by A. and L. Föppl, Vol. 2, Second Edition, 1928, p. 73.

[15] "Theory of Elasticity," by S. Timoshenko, 1934, p. 259.

formula; and, second, the writers had before them the detailed data obtained by P. A. Cushman[16] in tests with soap films. These data show clearly that the maximum shearing stress occurs at a point such as m in Fig. 27, and that there is a notable increase of stress from Point 1 to Point m. Special consideration was given to this increase.

Let τ_0 denote the shearing stress that would exist at Point 1 in Fig. 27 if the edge of the cross-section at that point were straightened out by moving the beginning of the fillet toward the left, when, at the same time, the thickness of the flange, t, the torsion factor, K, and the total twisting moment, T, are left unchanged. The stress, τ_0 is defined by the formula:

$$\tau_0 = \frac{T\,t}{K} \qquad \text{.....(57)}$$

Let τ_1 and τ_m denote the actual shearing stresses at Points 1 and m, τ_m being the maximum stress at the fillet. Then $\frac{\tau_1}{\tau_0}$ is the concentration factor at Point 1 and $\frac{\tau_m}{\tau_0}$ is the desired concentration factor for the fillet.

Prandtl's soap film analogy, which the authors have used advantageously, furnishes the key to the solution. It is noted that the soap film is stretched over an opening shaped like the cross-section and is inflated a small amount by an excess air pressure on one side. The shearing stresses on the original cross-section follow the contour lines on the film, the edge being one of the contour lines. Furthermore, the shearing stresses are proportional to the slopes of the film, or proportional to the density of the contour lines.

The contour lines at Section 1–2 in Fig. 27 must be approximately perpendicular to that section. Accordingly, the shearing stresses, τ, at that section are approximately in the direction of y. Since the slope of the film is $c\tau$, c being a constant, the curvature of the film at Section 1–2 in the direction of x becomes $c\frac{d\tau}{dx}$. The curvature of the film in the direction of y at Section 1–2 is accounted for by the curving of the contour lines. If the radius of curvature of the contour lines at a particular point is R, then the curvature of the film in the direction of y at the same point becomes $c\frac{\tau}{R}$. The curvature of the surface is the sum of the curvatures in the directions of x and y; that is, $c\left(\frac{d\tau}{dx} + \frac{\tau}{R}\right)$.

The equilibrium of the film requires that this combined curvature be constant. Since $R = r$ at Point 1 and $R = \infty$ at Point 2, it follows that,

$$\left[\frac{d\tau}{dx}\right]_1 + \frac{\tau_1}{r} = \left[\frac{d\tau}{dx}\right]_2 \qquad \text{.....(58)}$$

[16] "Shearing Stresses in Torsion and Bending by Membrane Analogy," by P. A. Cushman, Doctoral Dissertation, Univ. of Michigan, 1932.

Since Points 1 and 2 are on the same contour line, it is also required that,

$$\int_0^t \tau \, dx = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (59)$$

The diagram of the shearing stresses at Section 1-2 must be shaped about as shown in Fig. 28. Equation (60) has been constructed so that it satisfies this general requirement of shape as well as the specific requirements in Equations (58) and (59), and, therefore, it may be assumed to represent the shearing stress approximately:

$$\tau = \tau_1 \left[1 - \left(2 + \frac{t}{3r} \right) \frac{x}{t} + \frac{x^2}{2tr} \right] \quad \ldots\ldots\ldots\ldots (60)$$

The constant curvature of the film may be computed as:

$$c \left[\frac{d\tau}{dx} \right]_2 = - \frac{2c\tau_1}{t} \left(1 - \frac{t}{3r} \right) \quad \ldots\ldots\ldots\ldots\ldots (61)$$

If the fillet were some distance away, this curvature would remain the same, τ_1 would be replaced by τ_0, and the term containing r would disappear. Consequently,

$$\tau_0 = \tau_1 \left(1 - \frac{t}{3r} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (62)$$

which gives the concentration factor at Point 1,

$$\frac{\tau_1}{\tau_0} = \frac{1}{1 - \dfrac{t}{3r}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (63)$$

The important indications of Equation (63) are preserved when the following simpler formula is substituted:

$$\frac{\tau_1}{\tau_0} = 1 + \frac{t}{3r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (64)$$

As was mentioned, the maximum stress, τ_m, occurs at a point such as m in Fig. 27. Equation (64) can be used to obtain an estimate of this stress by making the following replacements: t is replaced by a distance, t_m, measured along a curved section as drawn from m to n; and τ_0 is replaced by a stress corresponding to straight edges and a thickness, t_m: that is, according to Equation (57), by $\tau_0 \frac{t_m}{t}$. Thus, the concentration factor becomes,

$$\frac{\tau_m}{\tau_0} = \frac{t_m}{t} \left(1 + \frac{t_m}{3r} \right) \quad \ldots\ldots\ldots\ldots\ldots (65)$$

A reasonable estimate of t_m is,

$$t_m = t + 0.3\, r \quad \ldots\ldots\ldots\ldots\ldots\ldots (66)$$

This value, substituted in Equation (65), leads directly to the formula represented graphically by the authors,

$$\frac{\tau_m}{\tau_0} = 1.2 + \frac{1}{3}\left(\frac{t}{r} + \frac{r}{t}\right) \ldots\ldots\ldots\ldots\ldots\ldots (67)$$

JOSEPH B. REYNOLDS,[17] ESQ. (by letter).[17a]—The problem of the twisting of I-beams has been studied thoroughly by the authors for special cases. It is the purpose of this discussion to compare the authors' results with a more general theory for the twisting of I-beams under differing types of loading. Only the effect of twisting is considered. The assumptions and the notation are the same as those used in the paper.

Consider the requirements for equilibrium at a section of the beam at a distance, x, from the fixed end as shown in Fig. 29. The inner moments at

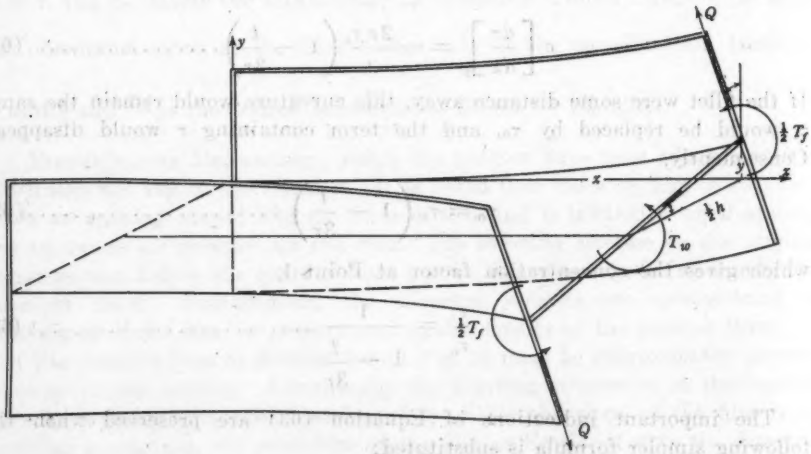

FIG. 29.

this section, together with the moment of the shears, must balance the outer moment applied to the beam to the right of the section. This outer twisting moment, T_x, will vary in general with the position of the section; that is, with x. For equilibrium at this section one must have:

$$T_f + T_w + Q\,h = T_x \ldots\ldots\ldots\ldots\ldots\ldots (68)$$

in which T_f, T_w, and T_x indicate twisting moments of flanges, web, and total of section, respectively.

In terms of the co-ordinates of points on the neutral axis, $y = f(x)$, the following approximate relation may be written:

$$\frac{1}{2}\,h\,d\theta = dy \ldots\ldots\ldots\ldots\ldots\ldots (69)$$

[17] Prof. of Math. and Theoretical Mechanics, Lehigh Univ., Bethlehem, Pa.

[17a] Received by the Secretary March 16, 1935.

and the twist per unit length is $\frac{d\theta}{dx}$, hence:

$$T_f + T_w = KG\frac{d\theta}{dx} = \frac{2KG}{h}\frac{dy}{dx} \quad \ldots\ldots(70)$$

From the relation between shear and bending moment:

$$\frac{1}{2}E_y I\frac{d^2y}{dx^2} = -Q \quad \ldots\ldots(71)$$

By means of Equations (70) and (71) and the value of a, Equation (68) can be written:

$$a^2\frac{d^2y}{dx^2} - \frac{dy}{dx} + X = 0 \quad \ldots\ldots(72)$$

in which, $X = \frac{h\,T_x}{2KG}$.

The differential in Equation (72), has the general solution:

$$y = A\sinh\frac{x}{a} + B\cosh\frac{x}{a} + C + y_1 \quad \ldots\ldots(73)$$

in which y_1 is the particular solution satisfying Equation (72) and the values of A, B, and C are determined by the boundary conditions for the beam. Equation (73) is the same as the authors' Equation (34) for the case considered. The value of y_1 is:

$$y_1 = \int X\,dx - \frac{1}{2}a\,e^{\frac{x}{a}}\int e^{\frac{-x}{a}}X\,dx - \frac{1}{2}a\,e^{\frac{-x}{a}}\int e^{\frac{x}{a}}X\,dx \quad \ldots\ldots(74)$$

For the usual types of loading, X can be written in the form:

$$X = X_0 + X_1 x + X_2 x^2 + \ldots + X_n x^n \quad \ldots\ldots(75)$$

in which, $X_0, X_1, X_2, \ldots X_n$, are constants. When Equation (75) holds successive integration by parts shows that Equation (74) may be written in the form:

$$y_1 = \int X\,dx + a^3\left(\frac{dX}{dx} + a^2\frac{d^3X}{dx^3} + a^4\frac{d^5X}{dx^5} + \ldots\ldots\right) \quad \ldots\ldots(76)$$

If n is even in Equation (75) the last term in Equation (76) is $\frac{d^{n-1}X}{dx^{n-1}}$; if n is odd, the last term is $\frac{d^nX}{dx^n}$. With the value of y_1 thus determined one may, by Equation (73) and the relation, $M = \frac{1}{2}EI\frac{d^2y}{dx^2}$, write:

$$M = \frac{2KG}{h^2}\left[A\sinh\frac{x}{a} + B\cosh\frac{x}{a} + a^2\left(\frac{dX}{dx} + a^3\frac{d^3X}{dx^3} + a^5\frac{d^5X}{dx^5} + \ldots\right)\right] \quad (77)$$

and, since the shear is given by $Q = \frac{1}{2} EI \frac{d^3 y}{dx^3}$:

$$Q = \frac{2 K G}{a h^2}\left(A \cosh \frac{x}{a} + B \sinh \frac{x}{a} + a^3 \frac{d^2 X}{dx^2} + a^5 \frac{d^4 X}{dx^4} + \ldots\right) \ldots (78)$$

The longitudinal stresses along the outer fibers of the flange will be given by $\sigma = \frac{M b}{I_y}$, or by:

$$\sigma = \frac{Eb}{2 a^2}\left(A \sinh \frac{x}{a} + B \cosh \frac{x}{a} + a^2 \frac{dX}{dx} + a^5 \frac{d^3 X}{dx^3} + a^7 \frac{d^5 X}{dx^5} + \ldots\right) \ldots (79)$$

and the lateral shearing stresses in the flanges by $\tau = \frac{Qb^2}{4 I_y}$, or by:

$$\tau = \frac{Eb^2}{8 a^3}\left(A \cosh \frac{x}{a} + B \sinh \frac{x}{a} + a^3 \frac{d^2 X}{dx^2} + a^6 \frac{d^4 X}{dx^4} + \ldots\right) \ldots (80)$$

The angle through which the beam twists comes from the relation, $\theta = \frac{2 y}{h}$, and has the value:

$$\theta = \frac{2 A}{h} \sinh \frac{x}{a} + \frac{2 B}{h} \cosh \frac{x}{a} + \frac{2 C}{h} + \frac{2}{h}\int X\, dx + \frac{2 a^3}{h} \times \frac{dX}{dx} + \frac{2 a^5}{h} \times \frac{d^3 X}{dx^3} + \frac{2 a^7}{h} \times \frac{d^5 X}{dx^5} + \ldots \ldots \ldots \ldots \ldots \ldots (81)$$

In this manner, general values are derived for the principal variables of interest in the case of twisted beams. The maximum values of these quantities will occur for differing values of x, depending upon the conditions surrounding the strained beam. Example (*a*) demonstrates that Equations (73) to (81) reduce to those given by the authors when the proper limitations are applied.

Example (a).—Beam Twisted by Constant Torque, T, Applied at Its Ends, with Both Ends Restrained.—In this case, $T_x = T$, a constant, and, therefore, $X = \frac{h T}{2 K G}$. The constants A, B, and C are determined by the requirements that for $x = 0$, $y = 0$, and $\frac{dy}{dx} = 0$. Furthermore, for $x = \frac{l}{2}$, $\frac{d^2 y}{dx^2} = 0$. Since the derivations of X are all zero, Equation (76) yields $y_1 = \frac{h T x}{2 K G}$ and, by Equation (73):

$$y = \frac{T h a}{2 K G}\left(\cosh \frac{x}{a} \tanh \frac{l}{2 a} - \sinh \frac{x}{a} + \frac{x}{a} - \tanh \frac{l}{2 a}\right) \ldots (82)$$

which is the same as that presented by the authors in Equation (35). The maximum displacement, y_m, occurs where $x = l$ and has the value given in the authors' Equation (36).

Similarly, Equation (77) reduces to the authors' Equation (37) for the moment in each flange, which has (where $u = \frac{1}{2}\,\frac{l}{a}$, or $x = 0$), a maximum value:

$$M_m = \frac{T\,a}{h} \tanh \frac{l}{2\,a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (83)$$

Equation (78) reduces to Equation (38) and Equation (79) becomes Equation (39).

By use of Equation (80):

$$\tau = \frac{T\,b^2}{4\,I_y} \frac{\cosh u}{\cosh \frac{l}{2\,a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (84a)$$

and,

$$\tau_{max} = \frac{T\,b^2}{4\,I_y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (84b)$$

which corresponds to the authors' Equation (42), and by Equation (81):

$$\theta = \frac{T\,a}{G\,K}\left(\cosh \frac{x}{a} \tanh \frac{l}{2\,a} - \sinh \frac{x}{a} + \frac{x}{a} - \tanh \frac{l}{2\,a}\right) \ldots (85)$$

from which,

$$\theta_{max} = \frac{T\,a}{G\,K}\left(\frac{l}{a} - 2 \tanh \frac{l}{2\,a}\right) \ldots\ldots\ldots\ldots\ldots\ldots (86)$$

Example (b).—Beam with One End Fixed, the Other Free, Under Constant Torque, T.—In this case, $X = \frac{h\,T}{2\,K\,G}$ as in Example (*a*). The conditions determining *A*, *B*, and *C* are that for $x = 0$, $y = 0$, and $\frac{dy}{dx} = 0$, and for $x = l$, $\frac{d^2y}{dx^2} = 0$. The value of the deflection obtained is:

$$y = \frac{T\,h\,a}{2\,K\,G}\left(\cosh \frac{x}{a} \tanh \frac{l}{a} - \sinh \frac{x}{a} + \frac{x}{a} - \tanh \frac{l}{a}\right) \ldots . (87)$$

From this all the other principal variables are readily found as in Example (*a*).

Example (c).—Beam with Both Ends Fixed and a Uniform Eccentric Load Along Its Length Producing an External Moment, T.—In this case, $T_z = T\left(1 - \frac{2\,x}{l}\right)$, and, $X = \frac{h\,T}{2\,K\,G}\left(1 - \frac{2\,x}{l}\right)$ giving,

$$y_1 = \frac{h\,T}{2\,K\,G}\left(x - \frac{x^2}{l}\right) - \frac{a^2\,h\,T}{K\,G\,l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (88)$$

The values of A, B, and C are determined by the requirements that for $x = 0$, $y = 0$, and $\frac{dy}{dx} = 0$, and for $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$. The deflection found is:

$$y = \frac{T h a}{2 K G}\left(\frac{\cosh u}{\sinh \frac{l}{2a}} - \coth \frac{l}{2a} + \frac{x}{a} - \frac{x^2}{a l}\right) \ldots\ldots (89)$$

Example (d).—Beam with One End Fixed and the Other Free, Uniform Eccentric Load Along Its Length.—Here, $Tx = T\left(1 - \frac{x}{l}\right)$ and,

$$y_1 = \frac{h\,T}{2 K G}\left(x - \frac{x^2}{2l}\right) - \frac{T h a^2}{2 K G l}$$

The constants, A, B, and C, are determined by the requirements that for $x = 0$, $y = 0$, and $\frac{dy}{dx} = 0$, and for $x = l$, $\frac{d^2y}{dx^2} = 0$. The deflection proves to be:

$$y = \frac{T h a}{2 K G}\left[\frac{\sinh\left(\frac{l}{a} - \frac{x}{a}\right)}{\cosh \frac{l}{a}} - \tanh \frac{l}{a} + \frac{a}{l}\operatorname{sech} \frac{l}{a}\left(\cosh \frac{x}{a} - 1\right) + \frac{x}{a} - \frac{x^2}{2 a l}\right] \ldots\ldots (90)$$

It is thus shown that the formulas developed and substantiated by test results in the authors' investigation, are identical with those obtained by a general mathematical treatment of the subject of torsional resistance.

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DISCUSSIONS

WATER POWER DEVELOPMENT OF THE ST. LAWRENCE RIVER

Discussion

BY DANIEL W. MEAD, HON. M. AM. SOC. C. E.

DANIEL W. MEAD,[20] HON. M. AM. SOC. C. E. (by letter).[20a]—The writer is pleased with Mr. Hogg's frank elucidation of the Canadian point of view in regard to the proposed St. Lawrence power development. He is in agreement with the views expressed far more than the discussion would seem to indicate. For example, he agrees with the first statement that "* * * the choice of a single-stage or a double-stage plan cannot be settled satisfactorily solely on engineering or economic grounds." This statement is verified by the official treaty lately before the Senate for consideration, which by no stretch of the imagination can be considered as the best engineering or economic solution.

The writer also agrees with Mr. Hogg that "* * * the fact that it [the St. Lawrence] is an international river cannot be ignored." While, in general, governmental activities must perhaps be based on "political" economy instead of on economics, the difficulties become greatly accentuated when a project becomes international and increase probably somewhat faster than the square of the number of Governments mutually interested in a project.

These difficulties practically vanish, however, if between two nations the administration of one of them is so anxious for the project to materialize that it is not only ready to submit to dictation as to the plans for the development and its cost, but also to surrender, forever, sovereignty over non-international waters lying entirely within its own borders, and, finally, to limit arbitrarily the use of such waters by its own people.

NOTE.—The paper by Daniel W. Mead, Hon. M. Am. Soc. C. E., was presented at the Joint Meeting of the Power Division of the Society with the American Institute of Electrical Engineers and the Hydraulics Division of the American Society of Mechanical Engineers, Chicago, Ill., June 29, 1933, and was published in August, 1933, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: August, 1933, T. H. Hogg, M. Am. Soc. C. E.; November, 1933, by Messrs. Theron M. Ripley, W. S. Lee, Frank E. Bonner, Rufus W. Putnam, James W. Rickey, Walter M. Smith, J. W. Beardsley, and L. F. Harza; and December, 1933, by Messrs. Adolph J. Ackerman, and L. F. Harza.

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[20a] Received by the Secretary January 14, 1935.

Canada and the Province of Ontario are to be congratulated on their intelligent handling of this entire project. To reiterate Mr. Hogg's statement (under the heading, "Reasons for Not Accepting Author's Set-Up"), "* * * the Canadian organizations studying the problem have been living with the river for a number of years, and * * * practically the same personnel has been engaged in the work throughout." The Canadian and Provincial organizations worked independently, but conferred and discussed the problem fully. They found the single-stage development of greater economic value, but such a project involved the raising of the water of the St. Lawrence for about forty miles up the river above the projected dam site. The Canadian inhabitants along this stretch of river apparently became alarmed at the proposal to increase the height of the water, which involved (to them) unknown danger, real or imaginary. Therefore, they appealed to their Parliamentary representatives, and apparently succeeded in creating so strong a public opinion as to render this plan politically impossible.

With the Crysler Island Two-Stage Plan, only the upper twenty miles of the St. Lawrence is to be raised, and this to a less degree; so that while the people along this reach also objected, the ill effect was not considered so apparent and their influence was not sufficient to block the demand of the Ontario Hydro-Electric Commission for the maximum amount of power which could be developed "all things considered." Having fully decided what they could and would do, the Canadian members of the Joint Board were prepared to meet the members of the Joint Board from the United States.

The policy of the United States in regard to the study of this project has been quite different. It has been the continuing policy of the U. S. Government, as far as the Corps of Engineers is concerned, to place an engineering officer on a Federal job for a limited time only, and about the time that he becomes really useful to send him to some other and often quite different type of project. On the power project of the St. Lawrence International Section, for example, there seems to have been no continuing individual service of the American representation; each of the American Sections of the several Joint Boards that have considered this subject, has consisted in general of entirely different engineering officers and, hence, these Sections have been at a great disadvantage and more or less dependent on the Canadian Section and the information furnished by it.

The arbitrary position of the Canadian Section of the Joint Board, and the substantial agreement of the Power Authority of the State of New York and its engineers with the writer's conclusions, is best described by reference to the Second Annual Report of the Power Authority of the State of New York for the year ending December 31, 1932 (pp. 17–18).

Referring to the complications introduced in the negotiations between the U. S. Department of State and the Power Authority by the question of the engineering plans for the development of the International Rapids Section, the report states:

"* * * the Power Authority early in the negotiations directed its engineers to make a thorough study of all the plans that had been proposed by the engineers of the United States and Canada.

"They reported that the plan prepared for the St. Lawrence Power Development Commission of New York by its Advisory Board of Engineers headed by the late Major General Edgar Jadwin was the most economical and efficient, from the standpoint of both power and navigation, that had been proposed.

"The Trustees therefore formally presented this plan to the State Department with a recommendation that it be given serious consideration by the Joint Board of Engineers which had been reconvened by the United States and Canada in connection with the treaty negotiations. The Power Authority was subsequently informed that the American members of the Joint Board strongly favored this single stage plan but that the Canadian members were insisting upon a two-stage plan with the upper dam at Crysler Island."

According to this report, the objections of the Power Authority to the Crysler Island Plan were laid before the State Department on January 6, 1932. Being strongly impressed, the representatives of the Federal Administration arranged for a joint conference with the representatives of Canada on January 16, 1932. At this conference the engineers of the Power Authority proposed a modified two-stage plan, with a main dam at Massena Point, and a control dam (at which no power could be developed) at Galop Island. It was hoped that this compromise might be acceptable because it was similar to a plan that had originally been preferred by the Canadian engineers and that was abandoned only because the United States Engineers objected to the relatively slight loss of power involved.

The Canadians rejected this plan, however, as well as all proposals for single-stage development and, to quote from the report:

"When it became clear that a treaty could not be negotiated that did not provide for a two-stage plan, the Power Authority adopted the position that, whatever plan might be agreed upon between the United States and Canada, the power development in which New York was primarily interested should not be required to pay more than the power project would cost if constructed in accordance with the most economical plan."

When the foregoing conditions are considered in the light of the American Section's final instructions from Washington (see paper under the heading, "Introduction"), and the rejection by the Canadian Section of any plan other than the Crysler Island Two-Stage Plan, the findings of the Joint Board of Engineers (Reconvened) that the two-stage plan "is practicable and feasible from an engineering point of view," seems the only one possible under the circumstances.

In some matters the writer is obliged to differ with Mr. Hogg. His statement (under the heading, "General") that "he [the writer] also shows a marked preference" for the New York Massena Point plan, conveys an impression that the writer was in favor of the plan. The writer knew nothing about the proposed New York plan until, in February or March, 1932, when he began a study of the various reports, referred to in the paper. The writer has no preference with respect to the different plans other than that which resulted from a critical analysis of those reports. He does not doubt that Mr. Hogg believes that the estimates of the New York Commission's report are on an entirely different basis from the two Canadian estimates of the two-stage and the single-stage plans, A and B, and that the corrections which he sug-

gests remove the Massena Point plan entirely from reasonable consideration. On the other hand he cannot believe that the late Maj-Gen. Edgar Jadwin, U. S. A., M. Am. Soc. C. E., the distinguished Chairman of the New York Commission (who, also, at one time served as Chairman of one of the American Sections of the Joint Board and who apparently was well informed on the subject) and the other distinguished civilian engineers who served with him on the New York Board, would furnish their client with an underestimate of more than $26 000 000.

As already explained, the writer is unable to answer specifically the questions raised by Mr. Hogg, and regrets that those civilian engineers who, being sufficiently well informed on this subject to have furnished an effective answer, preferred not to do so.

In the second paragraph under the heading, "General", Mr. Hogg explains what "accounts for what may appear to be lack of sufficient detail in the later reports." This lack is serious concerning not only the power development but the entire St. Lawrence project. Volumes have been written for and against the project, much of which has been in the nature of propaganda. There is—as far as the writer is aware—no place for the interested to turn to get an unquestioned fair and unbiased discussion or undisputed facts showing the desirability of the project, on which an intelligent opinion can be based.

Answering the comments of Mr. Ripley the writer would say that he was assigned the question of a comparison of the various plans of the proposed hydraulic power developments of the International Section of the St. Lawrence River and has not even attempted to discuss the economics of power transmission and use. Fortunately, this matter is covered at some length in the discussion of Mr. Bonner. Neither did the writer attempt to discuss the still greater and more complex question of the St. Lawrence Waterway in its broader significance. The latter question is so complicated that, apparently, little attempt has been made to justify the construction of this project which seems to have become largely a political matter. A fairly complete discussion of this subject appears in "The Great Lakes and the St. Lawrence Ship Channel," by R. S. MacElwee, Affiliate, Am. Soc. C. E., and A. H. Ritter, published in 1921. Based on an apparently impartial examination of the data available, and on a personal examination of the route by Mr. MacElwee, the authors of this volume conclude: (1) That both the waterway and the power development will be very profitable to the citizens of the United States and Canada; (2) that the saving on the shipping of grain will amount to from 8 to 10 cents per bushel; and (3) that great savings will be made on practically all that is produced within the area tributary to the Great Lakes. To quote from the book,

> "This saving will amount annually to approximately the entire cost of the improvement required to admit ocean vessels into the Lakes. * * * The water power developed will not only afford a revenue which will maintain the improvement and amortize the full cost within a reasonable period but it will be the direct means of creating an industrial development which in itself will provide a large traffic on the waterway. * * *"

On the other hand, the most thorough discussion of both the navigation and power questions with which the writer is familiar, is contained in the book entitled, "The St. Lawrence Navigation and Power Project," by Messrs. Moulton, Morgan, Lee, and others, published in 1929. This book is apparently disinterested and is based on the investigation of a number of people extending over several years. Their findings are:

First.—That the saving to the farmer will be 4 cents per bushel on the shipment of wheat and rye (which they estimate constitutes more than 50% of the probable total traffic) from Duluth, Minn., to Montreal, Que., Canada; that is, to reduce the costs from 9 cents per bushel (which was its 1929 price) to 5 cents per bushel over the St. Lawrence Waterway, will cost the taxpayer 11 cents per bushel, or involve a total cost directly and indirectly of 16 cents per bushel.

Second.—That while the power may be economical in connection with electro-chemical or electro-metallurgical industries situated on the river near the power plant, it will be uneconomical if transmitted to the New York Metropolitan, or to the New England, Districts.

Even if this opinion is correct, the project is apparently as sound as most of the Government navigation-power projects now under way. The economics of this problem are greatly involved and many opinions concerning them are diametrically opposed. In the writer's opinion this project ultimately will be carried to completion. If, in general, the internal navigation projects of the United States are at all warranted, this project which opens up a great territory of the United States and Canada to ocean navigation should ultimately be of great importance. It seems unfortunate that the nations involved have failed to investigate the economics of the problem to the fullest possible extent, but such investigations are so rare in Government undertakings that its absence in this case should not be surprising.

The writer appreciates the discussions of Messrs. Putnam, Smith, and Harza on the effects of these different power developments on navigation. This is a subject with which the writer is unfamiliar and the conclusions of these engineers indicate that the estimate, in the paper, of the greater value of the Massena Point Plant is not overdrawn.

The writer agrees with Mr. Ackerman that experimental models are a sound basis for determining the results which will be obtained from new hydraulic design and may well be utilized in some of the new problems in the proposed development of the St. Lawrence River. The difficulties involved in the question under consideration are not so much technical, however, as they are political. There is no question in the mind of the writer that the best plan can be made perfectly safe at an expense which, as compared with other plans, would fully warrant its adoption. The matter of convincing the electorate as to the questions of safety and advisability is entirely different. These are problems which quite possibly can not be demonstrated to the satisfaction of all concerned.

Canadian authorities recognize the fact that it is the part of wisdom to delay this project until the present depression is over. The following ex-

tract from an editorial in the *Canadian Engineer*[40] gives the point of view of at least one Canadian official:

"Last week further statements were made to the public regarding the St. Lawrence project. Thus C. Alfred Maguire, one of the Commissioners of the Ontario Hydro-Electric Power Commission, speaking regarding the Hydro-Power situation at Oakville, is reported as saying: 'I think Providence was with the Commission and the people of Ontario when they were not able to get the St. Lawrence at the particular time, because had we been able to get the St. Lawrence scheme completed we should have had $8 000 000 deficit every year. Instead of that the Commission got the private interests to build plants and give us the power from Quebec. * * *' Such a view expressed by one who is in close touch with these matters and who thus draws special attention to what might have proved a great economic loss, emphasizes the need for fuller economic consideration especially under the rapid and pronounced changes that characterize national and international trade relationships."

The writer agrees with Mr. Lee and Mr. Rickey that this important subject is a pertinent and proper subject for the discussion of engineers of both Canada and the United States and is worthy of the most careful, fair, and honest consideration.

[40] See Editorial, "Development in St. Lawrence Project," December 12, 1933.

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DISCUSSIONS

MODEL OF CALDERWOOD ARCH DAM

Discussion

BY A. V. KARPOV AND R. L. TEMPLIN,
MEMBERS, AM. SOC. C. E.

A. V. KARPOV[34] AND R. L. TEMPLIN,[35] MEMBERS, AM. SOC. C. E. (by letter).[35a]—Seven discussions were published applying to this paper, two of which came from abroad. The writers are much impressed with the exceptionally high character of each of these contributions. The different points developed in the discussion can be broadly summarized as follows: (I) The use of models in investigation of arch dam problems; (II) similarity conditions; (III) type of models; (IV) results of tests on Calderwood and other models; and, (V) design methods and theories. It will better answer the purpose of the closure to discuss each of these topics separately, rather than to consider individually the contribution of each discusser.

(*I*) *The Use of Models in Investigation of Arch Dam Problems.*—Two discussers, Mr. Janni and Dr. Probst, question the advisability of using models in the design of arch dams. Mr. Janni considers the temperature stress of utmost importance and does not believe that this stress can be measured on a model.

In a number of points the writers disagree with Mr. Janni. In so far as the theoretical computations are concerned, they are correct so long as the assumption can be made that there is a monolithic arch between fixed abutments on a non-yielding foundation. It seems that the usual simplified assumption of fixed abutments and a non-yielding foundation is quite firmly fixed in the engineering mind. For a number of structures such an assumption does not involve any considerable mistakes; but for an arch dam, consisting of a number of blocks between abutments, such simplified assumption leads to radically wrong conclusions.

NOTE.—The paper by A. V. Karpov and R. L. Templin, Members, Am. Soc. C. E., was published in December, 1933, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: April, 1934, by Messrs. A. W. Simonds and Lars R. Jorgensen; May, 1934, by A. C. Janni, M. Am. Soc. C. E.; and September, 1934, by Messrs. P. Wilhelm Werner, Eldred D. Smith, Elmer O. Bergman, and E. Probst.

[34] Designing Engr., Aluminum Co. of America, Pittsburgh, Pa.

[35] Chf. Engr. of Tests, Aluminum Co. of America, New Kensington, Pa.

[35a] Received by the Secretary March 14, 1935.

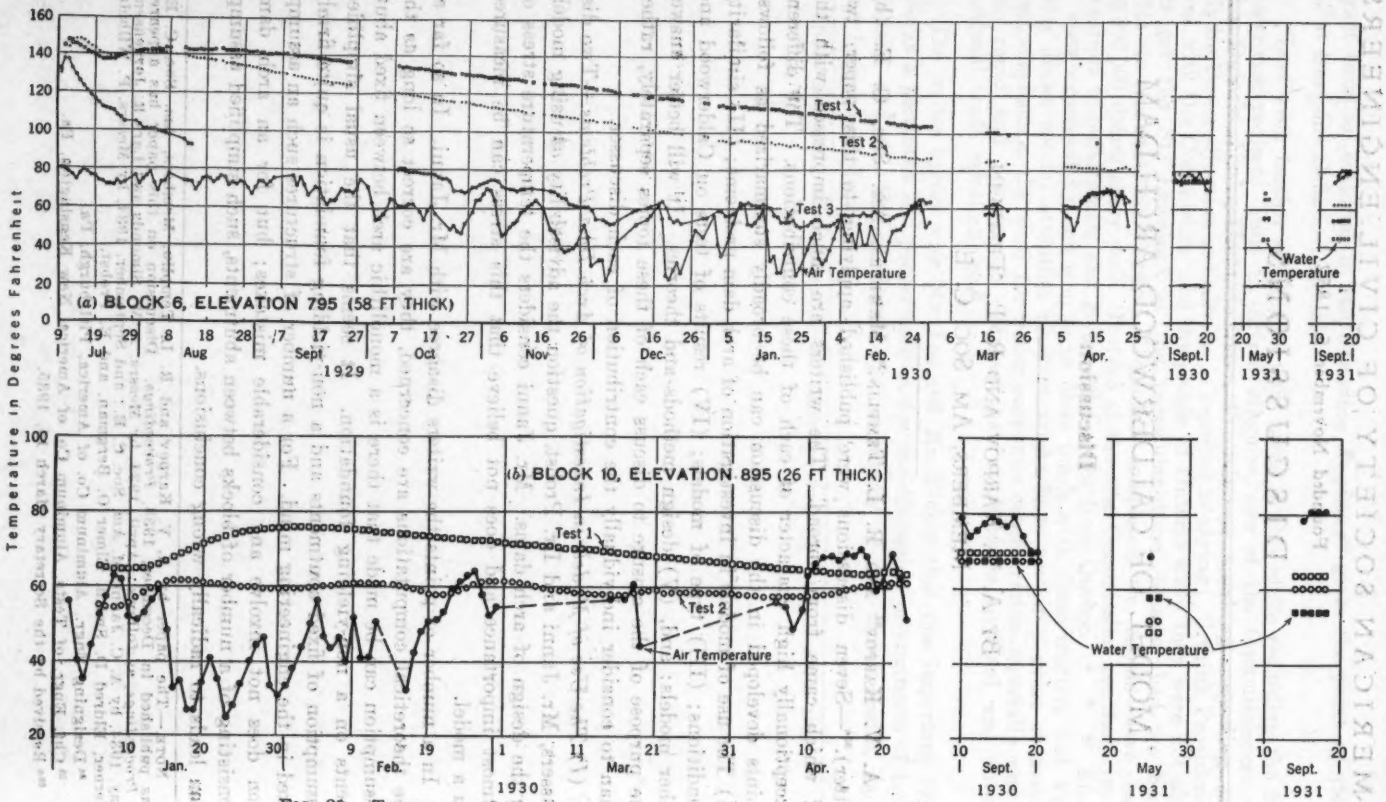

FIG. 21.—TEMPERATURE MEASUREMENTS, CALDERWOOD ARCH DAM; DAILY AVERAGES.

FIG. 21.—TEMPERATURE MEASUREMENTS, CALDERWOOD ARCH DAM; DAILY AVERAGES.

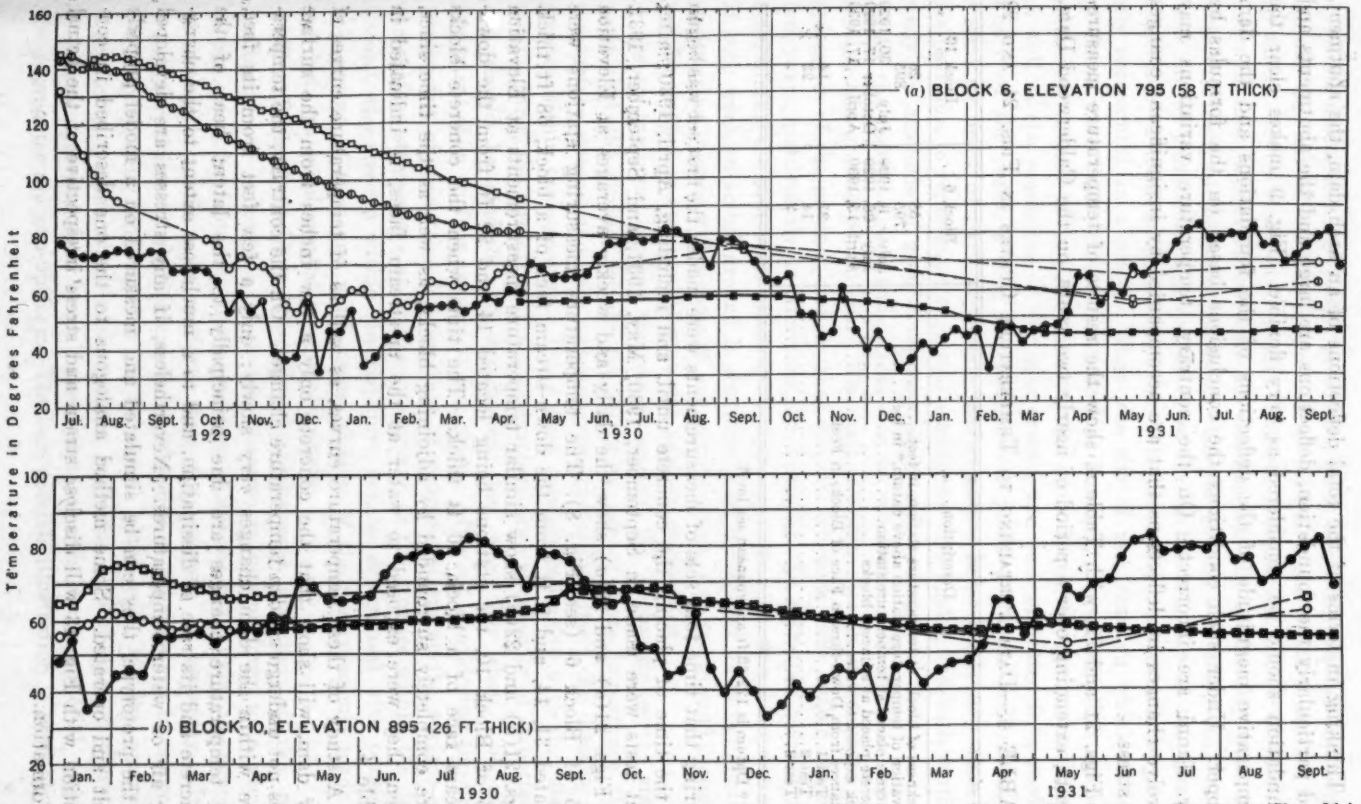

FIG. 22.—TEMPERATURE CURVES, CALDERWOOD ARCH DAM; WEEKLY AVERAGES. (FOR IDENTIFICATION OF CURVES SEE SIMILAR CURVES IN FIG. 21.)

Thinking in terms of the total deflections of an arch dam, the abutment, and particularly the foundation, deflections are large and the abutments and foundation should be considered as very flexible. Fig. 9 makes clear the comparative magnitude of the deflections of the foundations and the dam proper. Under such conditions the conclusions based on the formulas by Mr. Janni are incorrect. On the contrary, temperature variations may involve changes in deflections that are accompanied by insignificant changes in stress.

Figs. 21 and 22, with Table 4, show the results of temperature measurements, extending over a period of nearly two years, on the Calderwood Dam.

TABLE 4.—DATA PERTAINING TO TEMPERATURE CURVES IN FIGS. 21 AND 22

Description	Block 6	Block 10
Thickness of block, at temperature station, in feet	58	26
Elevation of temperature station above datum,* in feet	795	895
Concrete placed at temperature station	July 9, 1929	July 30, 1929
Concrete placed at adjacent blocks	July 26, 1929	October 17, 1929
Block exposed to water	April 15, 1930	April 17, 1930
Distances from Down-Stream Face of Block, in Feet:		
Test 1	31	14
Test 2	14	22
Test 3	2	..

* Datum is 122.62 ft above mean sea level.

During that time four sets of measurements were made; the first set was begun at the time of placing the concrete until, and including, April, 1930; after that tests were made in September, 1930, May, 1931, and September, 1931.

Figs. 21(*a*) and 22(*a*) show the daily and weekly averages at Elevation 795 of Block 6 (see Fig. 8). The temperature measuring stations were located 31, 14, and 2 ft from the down-stream face of a block, 58 ft thick. Figs. 21(*b*) and 22(*b*) show similar temperature measurements at Elevation 895 of Block 10, the stations being located 14 and 22 ft from the down-stream face of a block, 26 ft thick. The time when the concrete blocks were completely surrounded by adjoining blocks, as well as the time since, when they were exposed to water at the up-stream faces, is indicated in Table 4.

A study of these temperature curves, as well as of temperature curves of any dam, will show that the concrete only a few inches from the surface does not undergo sudden temperature changes. On the contrary, the temperature within the dam changes very slowly; only a few feet from its face, the temperature changes are due principally to the latent heat of the concrete and its rate of dissipation, and to a much less extent to the changing air or water temperatures. Nevertheless, if any stresses are developed in the prototype, they can be simulated and measured on a model properly built and operated. Some method analogous to the one described in connection with Fig. 13 will disclose strain and stress irrespective of the actual deformation.

In his discussion Dr. Probst presented, quite ably, the general arguments against the use of models in arch dam design. The writers are in full agreement with a number of points that are brought out in his discussion, although they cannot agree with his conclusions. In conformity with the ideas expressed in the paper he states that the ground and abutment conditions are of paramount importance in so far as the behavior of the dam is concerned. Furthermore, he declares that these conditions cannot be properly represented in a model. That was one of the ideas on which models of dams were built in the past; to-day, however, it is clear that such a statement cannot be justified because the ground and abutment conditions, as well as the conditions of the dam proper, can be represented in a model. Fig. 8 of the paper, giving the comparative model-prototype deflections, is a striking proof of that statement. Investigators know better now than in the past how to represent the ground conditions in a model, and with further study the methods of ground representation will be improved.

As the matter now stands, there are no particular difficulties in representing the ground conditions. The main difficulty lies in determining the properties of the ground; and, in that respect, the work done during the last few years, particularly in Germany, in applying the vibration method for determining the properties of the ground, seems to be of cardinal importance. If such work on actual ground is continued and is followed by model studies, engineering science will dispose of one more mystery, and the ground material will be treated as every other material used in engineering.

As an argument against the use of models, Dr. Probst makes the statement that the stresses within the structure are of more importance than the surface stresses and only measurements at the dam itself can disclose the stresses within the structure. The writers scarcely can agree with that statement. First, the stresses within the structure are undoubtedly of importance and should be studied and determined; but why they should be more important than surface stresses is not clear. After all, considering the practical point of view, the deterioration of concrete starts at the surface and the surface stresses are at least as important as the stresses within the dam. Second, the stresses within the structure can be determined with considerable saving of time and much more reliably, if the measurements made on the dam are supplemented by a model study.

Dr. Probst considers also the long-discussed question of the non-uniformity of concrete. Of course, concrete is non-uniform to a greater degree than, for instance, many commercial metals; but, again, the concrete of to-day is much more uniform than the concrete of yesterday. It is rather surprising to what extent the uniformity of concrete can be improved by proportioning and mixing in a central plant with rigidly controlled ingredients, proper water-cement ratio, and the liberal use of vibrators. Concrete aggregates in large masses act as a uniform material even if the aggregates are non-uniform. The best possible illustration is probably the

one taken from the ordinary commercially used metals. Every metal is composed of crystals that may have different elastic and pronounced directional properties. Nevertheless, so long as these crystals are distributed at random, but not preferably oriented, the material has a high degree of uniformity in pieces that are large when compared to the size of a single crystal. The elastic properties of such materials represent some average of the integrated values of the properties of the individual crystals. Proper proportioning, mixing, placing, and vibrating of the concrete will prevent any preferential orientation of the particles and in that respect a large mass of concrete probably is quite uniform, relatively speaking.

Dr. Probst arrives at the conclusion that since it is impossible to build a proper model of a gravity dam or an arch dam, no model tests should be attempted, but rather that measurements should be made on actual dams. The writers do not believe that measurements made on a prototype exclude the necessity of measurements on a model, or *vice versa*. From a theoretical viewpoint, measurements made on a prototype are the best. From a practical viewpoint, considering the magnitude of a dam, the inevitable limitations of such testing program and the impossibility of varying arbitrarily the loading conditions, such measurements are of relatively little importance and scarcely justify the time, work, and expense involved. These measurements, however, become of the utmost importance if accompanied by an adequate model study.

Taking as an illustration the measurements on the Bleiloch Dam mentioned by Dr. Probst: These measurements undoubtedly will give interesting results, but the practical application of these results is uncertain unless they are used in connection with a model study. Every investigation of this kind should disclose how improvements can be made and how safer dams can be built with the least possible material. Measurements made on a dam scarcely can give an answer to these questions unless supplemented by a model study.

Contrary to the opinions of the past, one cannot expect to use the results of measurements to develop an improved theoretical method of design. The question of to-day is not how to supplant one arbitrary set of assumptions by another just as arbitrary, but to find a method that will give results approaching the actual conditions and in this manner make possible an improved design.

Fig. 20 of Dr. Probst's discussion is of considerable interest, but does not support the contention that the stresses within the dam cannot be predicted from measurements on its faces. First, should be noted the unusual distribution (from the American viewpoint) of the construction joints. Second, the changes in the character of the temperature curves should be studied. Curves I, II, and III show the temperature before the dam was put in service and Curves IV, V, VI, and VII after water was stored above the dam. The breaks in these latter curves are an indication that water cir-

culated through the construction joints. Such circulation increased when the concrete blocks cooled and the size of the joints increased; this is manifested by the shape of the breaks of the temperature curves at the left construction joint shown on the diagram. If these joints were differently arranged and better grouted, no water would circulate through them and no breaks would have occurred in the temperature curves. Even under present conditions, however, after the concrete has cooled and the dam has attained its final service conditions, the breaks in the temperature curves should disappear. The regular shape of the temperature curve (Curve VII) at the right construction joint, is a rather convincing argument to that effect.

(*II*) *Similarity Conditions.*—There seems to be an agreement of opinion that the similarity conditions established in connection with the Calderwood model are complete and that a perfect model must meet every one of them. Mr. Smith is perfectly correct when he states that the writers took pains to prove the importance of the specific gravity requirements. It is rather discouraging, therefore, to the writers that this question is raised again, and attempts are made to minimize the importance of these conditions.

Considerable discussion comes from those who advocate the advisability of building an imperfect or only a partly correct model. In connection with such model, the relative importance of the different similarity conditions are discussed. Messrs. Simonds, Smith, and Bergman allege that the specific gravity requirements are of importance only if both the dead load and the live load stresses are measured, but that this requirement is of no importance if only live load stresses are measured and hence an imperfect model that neglects the specific gravity requirements is satisfactory for practical purposes.

The writers emphasize the dangers of the misapplication of the dimensional theory by the omission of a quantity that is difficult to handle, as was done by Professor Bergman in the derivation of his Equations (17) to (19). By using such a method, almost any desired statement can be proved and any improper similarity condition can be established. In order to apply the dimensional theory properly, all quantities entering into the problem must be considered. Only quantities that are negligible in their effect can be omitted without appreciable error. The omission of such an important quantity as the weight of the material of the dam will result in similarity conditions applicable to a weightless dam. Such similarity conditions could be used if a dam is investigated, the material of which has a very small specific gravity as compared with the loading liquid. These conditions, however, cannot be used with reference to an actual dam, the material of which has an appreciable specific gravity.

Fig. 18 of Mr. Smith's discussion and his elaboration on the subject as well as Equations (17) to (19) of Professor Bergman's discussion and his Fig. 19 are all based on the fallacious assumption that it is admissible to evaluate, separately, the dead load and live load stresses and to arrive at the total stress by simply adding these stresses.

A proper understanding of the more complicated three-dimensional stress problems is possible only if the principle of superposition is properly applied. The simple addition of the gravity and live load stresses would be permissible if a straight-line distribution of stress could be assumed. In addition to Fig. 1, Fig. 23,[36] shows clearly the incorrectness of the straight-line assumption.

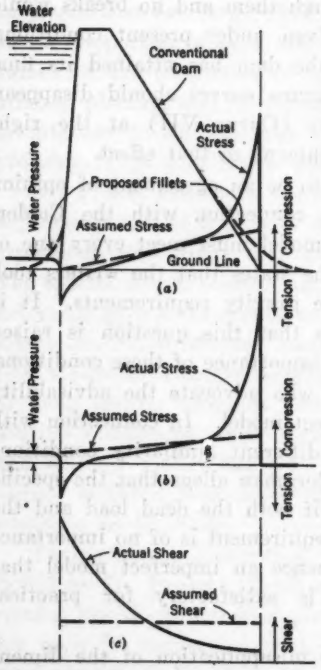

FIG. 23.—VERTICAL STRESS AND SHEAR AT THE GROUND LINE.

Professor Bergman states: "That a material with a Poisson ratio of 0.50 presents a special case." So far as a structural model is concerned, only a practically non-existent material with a Poisson ratio of zero could be considered as a special case. In such a material the application of stress in one direction would not produce strains or stresses in any other direction.

Due to the application of stress in one direction, any material having a Poisson ratio of 0.15 or 0.50 will induce strains or stresses in other directions, the only difference being in the magnitude of the strain or stress. In that connection it would seem that Professor Bergman's statement, "for such a value of the Poisson ratio [0.50] the material deforms with no change in volume, so that if it is compressed at one point, it must expand at another," is unfortunately worded, because there is no difference in the behavior of the material having a Poisson ratio of 0.15 or even 0.50. There is no question that "it must expand," but in both cases it will have the tendency to expand; if restrained from expansion both materials will develop stresses similarly, the only difference being in the amount of expansion and in the value of the induced stresses.

All the discussers who stress the importance of Poisson's ratio and the non-importance of the specific gravity requirements apparently overlook the importance of the proper representation of the foundations, abutments, and joints in the model. Mr. Smith even goes so far as to make the statement that the representation of joints in a model is undesirable and advocates the building of models that intentionally violate this similarity condition. All of which brings the entire problem back to the fundamental question : What is a modern structural model?

(*III*) *Type of Models.*—There seems to be a considerable difference in opinion as to the purpose of structural models and what constitutes an up-to-date engineering model. It should be realized that present engineering

[36] "Stability of Straight Concrete Gravity Dams," by D. C. Henny, M. Am. Soc. C. E. Discussion by A. V. Karpov, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 99 (1934), Fig. 12, p. 1083.

knowledge is limited[37] and that in many structures a wide discrepancy exists between the actual stress in the structure and the stress evaluated by the use of conventional methods. This discrepancy is very pronounced in statically complicated structures, the design of which is based upon assumptions that were satisfactory in the design of simple structures. Consequently, a rapid change in the conception of the theory of stress is currently taking place. Many assumptions that were considered axiomatic a few years ago are now not only challenged but often discarded.

The major problem of modern engineering is to determine the correct design assumptions which will suit the particular structure under consideration. The next in importance is the selection of a proper mathematical treatment which will make it possible to evaluate the different phases of the behavior of the structure. The limitations of present mathematical knowledge makes it impossible to obtain a complete and exact solution of the differential equations set up by introducing the theory of stress. Consequently, the strains, deflections, and stresses can be evaluated only approximately. The discrepancy between the actual and the theoretically determined behavior of a structure depends upon the correctness of the design assumptions and upon the approximations involved in the particular mathematical method used.

A correctly designed model has the important advantage that it eliminates the approximations of the mathematical methods and gives results that depend only on the correctness of the design assumptions, thus making possible an independent solution of the major engineering problem.

The uncertainties regarding some of the conditions of an arch dam, and particularly with respect to the foundation conditions, are used by Mr. Simonds as an argument against the use of models and in favor of the substitution of a mathematical analysis as the basis of design. It should be realized that the same uncertainties that enter into the design, the building, and the testing of a model, also enter into the mathematical analysis of the prototype. If, for instance, the actual ground conditions are not known, some assumptions regarding these conditions must be made and applied equally to the mathematical analysis of the prototype and to the building of the model. Any reasonable design assumption can be represented in an adequate model; but there is this radical difference between the mathematical analysis and the model study: The uncertain assumptions upon which the mathematical analysis is based will be imposed upon the ground assumptions which are used in the mathematical analysis. Due to the uncertainty of the ground assumptions and of the mathematical method the final results will differ from actual conditions. The part of the discrepancy that is due to the uncertain ground assumptions may be small when compared with the part that is due to the currently accepted mathematical methods. On the other hand, the same ground assumptions, if properly incorporated in an otherwise correct model, will be the only uncertainty, and the discrepancy between the actual

[37] "The Use of Engineering Models," by A. V. Karpov, M. Am. Soc. C. E., *The Military Engineer*, November–December, 1934.

behavior of the prototype and the predicted behavior, based on the model study, will depend only upon the correctness of the ground assumptions.

It was not the intention of the writers even to suggest the elimination of the mathematical analysis. The results of proper model studies can, and should, be used to develop and check a reliable design theory which could be finally incorporated in a mathematical design method. The writers contend that a mathematical theory which is checked by an incorrectly designed model cannot be considered as satisfactory; only a theory that can be checked on a correctly designed model may be considered satisfactory and hence should be used.

By the use of models there are two methods of approach to the major problem. The first method, often used in the past, is to make a set of assumptions which may appear to be either more simple or more reliable, and thereby attempt to obtain proof of the more complicated or less reliable assumptions. In an arch dam investigation numerous assumptions of this type are possible. The mercury loaded plaster-celite model of an arch dam is based on a series of such assumptions, some of which are: (*a*) That the total stress can be determined from the summation of the dead load stress, determined without considering the hydrostatic pressure, and the live load stress, determined under zero weight conditions; (*b*) that the behavior of an arch dam model with fixed foundations and abutments is similar to that of a prototype arch dam with flexible foundations and abutments, so that tests made on a model with fixed foundations and abutments will determine the condition of a prototype built on flexible foundations and abutments; and (*c*) that a dam having joints will behave similarly to a dam that is built as a monolithic structure. This method is a simple one. The assumptions can be chosen so that no particular attention need be paid to any similarity conditions. The mercury loaded models are usually scalar representations of the prototype; but even that is not necessary and an arbitrarily shaped model could be used. However, it should be realized that this method does not give any proof of the accuracy of the assumptions made; it merely substitutes one more or less arbitrary set of assumptions for another.

The second method is based on the omission of all arbitrary assumptions. This is accomplished by the determination of the similarity conditions that will establish a natural relation between the behavior of the model and its prototype. By building a model that satisfies these similarity conditions and its testing, the actual behavior of the prototype can be determined and the correctness of the design assumptions can be checked. A concise definition of such an up-to-date engineering model is given by Mr. Arthur C. Ruge,[33] who states:

"The true criterion is that it must be possible to deduce the behavior of the prototype, under an arbitrary set of conditions, from the behavior of the model when subjected to a similar and properly proportioned set of conditions."

[33] "The Determination of Earthquake Stresses in Elastic Structures by Means of Models," by Arthur C. Ruge, *Bulletin*, Seismological Soc. of America, Vol. 24, No. 3, July, 1934.

Some of the discussers prefer mercury-loaded models because they consider the high value of Poisson's ratio of the rubber-litharge compound a serious drawback of the water-loaded rubber-litharge model.

Table 5 permits a comparison with respect to the specific gravities of the model materials used in the three kinds of models referred to in the discussions.

TABLE 5.—SIMILARITY CONDITIONS WITH RESPECT TO SPECIFIC GRAVITY OF THE MODEL MATERIALS

Kind of model	Specific Gravity		Ratio of actual to required weight (percentages)
	Required	Actual	
Mercury-loaded plaster-celite	32.84	0.7	2.1
Mercury-loaded concrete	32.84	2.4	7.3
Water-loaded rubber-litharge	2.4	2.4	100.0

Comparing the mercury-loaded, plaster-celite model with the water-loaded, rubber-litharge model, it should be realized that the comparison is to be made between a model having only 2% of the required weight and without a proper representation of the foundations, abutments, and joints, and a model having 100% of the required weight and built of a reasonably uniform material, with properly represented foundations, abutments, and joints, but having a deficiency in Poisson's ratio. The difference between these two kinds of models can be visualized clearly by comparing three dams of the same size—an actual dam and two imaginary dams. The first imaginary dam is assumed to be built as a monolithic structure, on foundations and abutments that are very many times as stiff as the actual foundations and abutments, and using a material that has only 2% of the weight of the concrete of the actual dam. The second imaginary dam is assumed to be built in exactly the same way as the actual dam, on the same foundations and between the same abutments, using a material that has the same properties as the concrete of the actual dam, except the high value of Poisson's ratio.

The behavior of the first imaginary dam would be radically different from that of the actual dam, and it would be impossible to establish a rational inter-relation between the respective stresses and deflections of this imaginary dam and the actual dam. The behavior of the second imaginary dam would be substantially the same as that of the actual dam, the only difference being due to the different Poisson ratios. Applying secondary corrections, that will be necessary to account for the difference in Poisson ratios, it would be possible to compare directly the respective stresses and strains of the second imaginary dam and the actual dam.

In comparing the plaster-celite and rubber-litharge models a number of statements were made in the discussion concerning the uniformity of the plaster celite and the non-uniformity of the rubber litharge. There are no supporting data to confirm the statements to the effect that the plaster-celite

material is uniform. It is known[39] that this material is unstable in so far as its elastic properties are concerned. The presence of moisture in the model material, or high humidity of the surrounding air may result in pronounced plastic instead of the presumed elastic properties. Such qualities make the conditions uncertain. The statements regarding uniformity should be supported by results of tests properly made under conditions and on specimens that would assure that the test results are representative in so far as the material in the model itself is concerned.

The rubber litharge used in the Calderwood model was reasonably uniform. In so far as statements concerning the non-uniformity of the rubber-litharge compound of the Boulder Dam model are concerned, it should be kept in mind that this compound requires special technique in testing because if tested by conventional methods it is quite probable that incorrect results will be obtained.[10] The maximum value of Poisson's ratio of 0.64, stated by Mr. Simonds, makes the writers somewhat doubtful as to the reliability of the procedure and testing methods used.

The statement by Professor Bergman that "the chief advantage of rubber as a model material is that it enables strain measurements to be made on the up-stream face" is incorrect. The possibility of using strain-gauges on the water-covered face of the model is merely an incidental advantage, but it cannot be termed the "chief advantage." The chief advantage in using the rubber-litharge material is that measurements made on a water-loaded model will enable the prediction of the behavior of the prototype with satisfactory accuracy. No such predictions are possible where the measurements are made on a mercury-loaded plaster-celite model.

For future models it is very desirable to use a material that has a low modulus of elasticity, a specific gravity of 2.4, and a Poisson ratio between 0.15 and 0.20.[39] It would also be desirable to have a more uniform material, that is, with less than $\pm 15\%$ variation in the modulus of elasticity. Such refinements would improve future model work.

One of the most significant features of the work done on Calderwood Dam and its model is the proof of the important influence that the behavior of the ground and abutments exerts on the behavior of an arch dam. An arch dam cannot be considered as an isolated independent structure, but must be investigated as an integral part of the surrounding canyon. Considering the present status of the art, any model that neglects the proper representation of the foundation and abutments cannot be considered as an up-to-date model.

Mr. Simonds asks a number of pertinent questions regarding the representation of the foundations in the Calderwood model. It was stated in the paper that the investigation of the foundation conditions was necessarily one of the objectives. In studying the Calderwood model, the writers had at

[39] "Materials for Use in Structural Engineering Models, with Special Reference to Rubber Compounds," by A. V. Karpov, M. Am. Soc. C. E., *Proceedings*, Am. Soc. for Testing Materials, Vol. 34, Pt. II, 1934.

[10] "Methods for Determining the Physical Properties of Certain Rubber Compounds at Low Stresses," by R. L. Templin, M. Am. Soc. C. E., and R. G. Sturm, Assoc. M. Am. Soc. C. E., *Proceedings*, Am. Soc. for Testing Materials, Vol. 31, 1931.

their disposal the measurements made on the prototype and could adjust the foundations and abutments accordingly. If such measurements were not available, or in case of the building of a new model on a foundation for which sufficient information is not available, the writers probably would attempt to determine the behavior of the model in a number of steps, each of which would disclose the behavior of the dam under different flexibility conditions of the foundations. Having at their disposal the results of the tests on the Calderwood model, it is assumed that for any other prototype, model conditions could be obtained that would closely approximate the actual ones.

Professor Bergman's discussion as to the possible difference between the stress in a beam having the same deflections, but different loadings, is perfectly correct. However, under the conditions of a model having the proper specific gravity ratio, an additional condition must be introduced; that is, the impossibility of arbitrarily varying the loads. If that is taken into consideration the conclusions based on the example of the action of a straight beam will be quite different.

Referring to Mr. Werner's inquiry concerning the temperature control, the writers had in mind to utilize the low thermal conductivity of the rubber-litharge compound. If and when the testing of the model is resumed, it is planned to conduct a number of tests keeping the room temperature high for a period of time sufficiently long to have the temperature of the model material uniformly high. Next a series of tests should be run with lower room temperature and with a considerably lower water temperature. It seems that such a method of testing will result in a series of deflection and strain readings that will correspond to different temperature conditions of the dam, air, and water. The low thermal conductivity of the rubber-litharge compound should make it possible to obtain distribution of temperatures throughout the model quite similar to that of the prototype.

So far as Mr. Smith's work is concerned, the writers again point to the danger of the indiscriminate use of models. To build a model of a material that weighs only 42 lb per cu ft and, by applying external weights, increase its total weight about forty-nine times, or to about 2 050 lb per cu ft, would seem to indicate a serious violation of fundamental similarity conditions. At present, such models cannot be justified, because the procedure is contrary to modern ideas regarding correct model design and particularly since materials of proper specific gravity for use in water-loaded models are now readily obtainable.

(*IV*) *Results of Tests on Calderwood and Other Models.*—The results of tests on the Calderwood model differ from expectations based on conventional design ideas and also from results of tests made on mercury-loaded models.

The results of measurements of the vertical stress on the up-stream and down-stream faces of the model are probably the ones that raised the most questions. These questions cannot be answered under conventional assumptions; but if these assumptions are disregarded and if it is conceded that there is no straight-line distribution of stress and that the total stress is not

equal to the sum of the dead load and live load stress determined separately, the results of these measurements are very logical. The application of the live load results in a redistribution of the dead load stress; the character of this redistribution may be rather complicated on account of the influence of the foundations and abutments.

The present design methods do not disclose such redistribution of stress; neither can a model in which the weight of the material is subdued, and the weight of the loading fluid considerably exaggerated, show the redistribution of stress that will correspond to the actual conditions in the prototype.

Mr. Werner gives a convincing answer to the doubts raised with reference to the possibility of having vertical tension stress on the down-stream face of the dam. The instance of a cylindrical vessel cited in his discussion could not be improved upon by the writers.

Mr. Simonds refers to the measurements on the up-stream face of the model. It is only natural that there should be a difference in measured strains if pins of different lengths are used. Since the measurements should give the strains at the face of the dam, the shortest possible pins should be used. Indeed it may be advisable to substitute for the pins some device that would make it possible to attach the gauge points directly to the surface of the model.

Mr. Jorgensen's conclusion that the vertical elements of the dam impose additional loads on the upper arches and reduce the loads on the lower arches is correct. In so far as the deviation of the principal stresses from the vertical-horizontal directions is concerned, Fig. 10 shows that such a deviation is much smaller than is usually assumed and is pronounced only near the abutments. For dams that do not have a considerable increase of thickness toward the abutments, the deviation of the principal stresses from vertical-horizontal directions will be still less. Mr. Jorgensen believes that the assumption customarily made in designs neglecting the effect of gravity—that the horizontal arches slide frictionless between the adjoining arches—is on the side of safety. The writers would like to state that in the design of a complicated structure the only safe procedure is to base the design on correct assumptions and to eliminate all incorrect assumptions. If in such designs incorrect assumptions are made, it is impossile to foresee the complications that may arise, and it is very unlikely that an incorrect assumption can be made which will bring the design on the side of safety. The effect of gravity was neglected in the design of Calderwood Dam. This assumption, which was supposed to be on the safe side, resulted in the increased stiffness of the dam toward the abutments in the lower elevations. Fig. 11 shows clearly that this stiffness is responsible for the increase of stress between Blocks 6 and 7 at Elevation 860. Such increase of stress, of course, cannot be considered as making the dam safer.

Mr. Simonds' Equations (2) and (3) are correct only if the vertical and horizontal stresses are, at the same time, the principal stresses. If that is not the case, more complicated expressions should be used.[40]

[40] "The Determination of Stresses from Strains on Three Intersecting Gage Lines," by William R. Osgood and Rolland G. Sturm, Assoc. Members, Am. Soc. C. E., *Research Paper No. 559*, National Bureau of Standards, *Journal of Research*, Vol. 10, May, 1933.

Professor Bergman questions the method used to compare the stresses that were measured on the prototype and the model. Figs. 11 and 12 speak for themselves and show a reasonably close agreement between these stresses. The writers do not contend that this is the correct method but, as stated in the paper, believe that these diagrams are "indicative at least."

The writers are very much impressed by the two model tests described in Mr. Smith's discussion. First, a complete plaster-celite model of an arch dam was loaded with mercury. The deficiency in weight of that model was 98 per cent. The deflections of the crown cantilever were measured. Second, a slice model at the crown cantilever was built using the same plaster-celite material, but loading it by additional weights to about forty-nine times the weight of the original material. This model was loaded, not with mercury, but by using a wire pulley and weight arrangement, so as to have the load correspond to the mercury load that was carried by the cantilever. These loads were evaluated by the trial-load method. The cantilever deflections in both cases were found to be practically the same. To the writers the procedure seems wholly inadequate and was made necessary on account of the type of model used.

Nevertheless, Mr. Smith arrives at the conclusion that the trial-load method is correct and that the weight of the model material is of no importance. Actually, the deflections of the cantilever depend on the ground conditions and on the weight of the material of the dam. There does not seem to be any rational inter-relation between the two models and between these models and the prototype.

(*V*) *Design Methods and Theories.*—The different methods used in the design of arch dams are summarized in a paper by one of the writers.[41] Typical cross-sections are given[42] of a number of dams designed by the two design methods that are most used at present. One design neglects the gravity action and the other uses the trial-load method in which an attempt is made to consider the gravity action.

The measurements made on Calderwood Dam and the model work described in the paper seem to indicate that the neglect of the gravity action leads to a radical discrepancy between the actual and the design conditions. From the current engineering point of view such a method cannot be considered satisfactory. The trial-load method represents some improvement.[42] However, this method is far from being satisfactory or even reasonably approaching a method that should find general recognition.

A tendency seems to be developing in arch dam design to consider everything highly satisfactory and to disregard any evidence to the contrary. The supporters of the trial-load method seem to be going very far in that direction. It may be of interest to survey the present conditions. More or less satisfactory measurements were made on two commercial dams: Calderwood Dam, on which the most extended series of deflection and stress measurements were made, and Ariel Dam, on which deflection measurements were made, but

[41] "The Compensated Arch Dam," by A. V. Karpov, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 1309.

[42] *Loc. cit.*, Figs. 1 and 2, pp. 1310–1311.

stress measurements were omitted. On the Gibson Dam preparation for making measurements was included, but unfortunately no measurements of value could be taken. The measurements made on Calderwood Dam could not be verified by the use of the "trial-load" method. The Ariel Dam was designed by the use of the trial-load method and the deflection measurements disclosed a rather startling difference in the evaluated and measured deflections.[43]

The only tests by which the trial-load method could be verified are the model tests made on mercury-loaded models on which the specific gravity requirement was grossly violated and on which neither the foundation and abutment conditions nor the joint conditions were properly represented. Under such conditions it seems that the statements so often made in recent years, that the designs of arch dams are based on proven test data, are overoptimistic. It is no secret to the Engineering Profession, that the safety of high arch dams is not obtained on account of sufficient engineering knowledge, but rather to the contrary: The present limited engineering knowledge makes it necessary to use in such dams a much larger amount of material—unsatisfactorily distributed—than would be necessary if the design could be based on proved assumptions. In that respect the horizontal cracks or the openings of horizontal construction joints in arch dams are very significant. In the latest designs, and in particular in the trial-load designs, vertical tension stress that could produce cracks or joint openings of such character presumably are absent; consequently, the tendency is to attribute such cracks to faulty construction methods, contraction of the concrete, etc. In fact, such an explanation was accepted a few years ago on one of the arch dams with which the writers were connected. The tests on Calderwood Dam and its model show very clearly that such cracks, not caused by faulty construction methods or by shrinking concrete, may be caused by high vertical tension stress at both the up-stream and the down-stream face of the dam. Such stresses cannot be evaluated by using one of the conventional design methods; nor can they be disclosed on mercury-loaded models.

The proper design of an arch dam should give consideration to the following items: (1) Flexibility of foundations and abutments; (2) non-linear distribution of stress; (3) substitution of properly shaped horizontal arches instead of the arbitrarily chosen circular arches; (4) influence and proper utilization of the construction joints; and (5) necessity of liberal use of fillets.

These items can be investigated on a model in which the specific gravity requirements are satisfied. A low value of Poisson's ratio would make a more satisfactory model, but even when using a material with a high Poisson ratio satisfactory conclusions can be drawn.

The mercury-loaded plaster-celite model does not make possible the proper investigation of any of these most important requirements and, consequently, no satisfactory design theory can be developed from studies on such models.

[43] "Actual Deflections and Temperatures in a Trial-Load Arch Dam," by A. T. Larned and W. S. Merrill, Members, Am. Soc. C. E.; Discussion by A. V. Karpov and R. L. Templin, Members, Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 951.

Conclusion.—In conclusion, the writers would like to emphasize a few points which often seem to be neglected, but which are of considerable importance in the model study of structural problems.

1.—It should be realized that the new conceptions in physical and chemical sciences necessarily involve engineering. Designers of important structures should take cognizance of the fact that the conventional stress theories do not represent the actual conditions, and that, in the building up of the new theories, many engineering assumptions must be revised and many rejected, notwithstanding the fact that until recently they were considered perfectly sound.

2.—The present design methods do not disclose the importance of obtaining reasonably flexible structures. No matter how massive or stiff an engineering structure may be, it acts substantially as an elastic structure. The applied load introduces stresses that result in strains, deformations, and deflections. The value of the actual stress can be determined accurately if very refined design methods, based on proper assumptions, are utilized. Very often, due mostly to lack of knowledge, less refined design methods combined with unreliable assumptions are used. Then the actual stresses, at least in some parts of the structure, will be higher than the evaluated stresses; and stress concentrations that were not foreseen in the design, will occur.

Comparing two similarly designed structures of a different degree of flexibility, the stress concentrations in a more flexible structure will be not as high as in a stiffer structure, the more flexible structure having a better opportunity to distribute the stress more uniformly.

In some cases an increase in material in some parts of the structure may increase considerably the stiffness and thus result in much higher stress concentrations. Under such conditions, contrary to the design data, the actual stress may be higher in some parts of a structure in which more material is used as compared with a structure in which less material is used. Designs obviously should not be used in which the stresses are increased on account of the use of an increased amount of material.

3.—A properly designed model, if used discriminately, is one of the most valuable of engineering tools and permits the correct solution, among others, of the most complicated structural problems; but models are not miraculous tools, and valuable results are obtainable only if theoretically correct models are used.

4.—In any model study it is necessary to approach the problem without preconceived ideas to be proved or confirmed. Only a model which is given a fair chance to act naturally and in a manner similar to the prototype will give reliable results. Models that are forced to act in a certain way dissimilar to the action of the prototype, in nearly all cases will be misleading, and should not be used.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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DISCUSSIONS

AN APPROACH TO DETERMINATE STREAM FLOW

Discussion

BY MERRILL M. BERNARD, M. AM. SOC. C. E.

MERRILL M. BERNARD,[42] M. AM. SOC. C. E. (by letter).[42a]—Since April, 1934, the writer has been associated with a group in the U. S. Geological Survey which has been engaged on a study of floods and flood frequencies, and rainfall, run-off, and inter-related factors. A large part of his time has been devoted to the application of the unit-hydrograph method to run-off resulting from storms of high magnitude. Underlying principles and methods have been studied by other members of the group, as outlined in the discussion of Messrs. Hoyt, Harrold, and Snyder. In the preparation of his closing discussion, the writer has drawn on his own additional experience as well as on the ideas and studies of his associates.

The simplified form of Equation (1), suggested by Mr. Jarvis in Equation (5), should be accceptable, in general, particularly in cases where channel characteristics are not known. The factor, U, is purely an abstract one, having its principal use in the construction of regional charts, as shown in Fig. 2.

The suggestion that the pluviagraph be used as a guide to the shape of the hydrograph has possibilities. Recent studies indicate that, where only definite flood hydrographs are considered, the ratio, $\frac{\Sigma \text{ hydrograph}}{\Sigma \text{ pluviagraph}}$, for any basin follows a rather well-defined trend throughout the seasons. Such a trend curve can be developed from a comparatively short record of flow, and would be a more dependable guide to run-off coefficients for a particular locality within the Mississippi Basin than the range of from 14 to 31%, suggested by Mr. Jarvis.

NOTE.—The paper by Merrill M. Bernard, M. Am. Soc. C. E., was published in January, 1934, *Proceedings*. Discussion in this paper has appeared in *Proceedings*, as follows: March, 1934, by C. S. Jarvis, M. Am. Soc. C. E.; April, 1934, by LeRoy K. Sherman, M. Am. Soc. C. E.; May, 1934, by W. W. Horner, M. Am. Soc. C. E.; September, 1934, by Messrs. C. H Eiffert and Charles S. Bennett; November, 1934, by R. L. Gregory and C. E. Arnold, Assoc. M. Am. Soc. C. E.; and January, 1935, by Messrs. Gordon R. Williams, and W. G. Hoyt, L. L. Harrold, and F. F. Snyder.

[42] Cons. Civ. Engr., Crowley, La.

[42a] Received by the Secretary March 4, 1935.

The writer is in accord with Mr. Sherman on the points covered by his discussion. Particularly does he join him in concurring with Mr. Horton and others in the belief that infiltration loss should be considered as a deduction from daily rainfall. It has been the writer's experience, however, that average or gradually decreasing infiltration deductions throughout periods of continuous or intermittent rainfall do not produce net rainfalls that can be converted directly into the stream hydrograph with the use of the distribution graph. For storm periods consisting of two or three days of appreciable rainfall, preceded by several days of light rainfall, all such light rainfall, together with the greater part of the first day of heavy rainfall, may have been consumed by infiltration, whereas the following days of heavy rainfall largely account for the resulting stream flow. It is also believed that the rainfall-recording interval of 24 hr obscures comparatively long periods between actual rainfalls which have been recorded as continuous, making the practical correlation of a daily infiltration deduction with antecedent conditions difficult.

The phenomenon of infiltration, which is the principal loss to rainfall, is so unlike that of surface run-off, that the two should not be directly combined to give the total loss. Infiltration activity is more logically associated with that of rainfall. Like rainfall, it can be applied to a unit of area and is practically accomplished within the duration of rainfall, whereas run-off, very shortly after it is created, is to be associated with surface and channel storage, and stream flow. In the case of a unit-time rainfall, producing a unit hydrograph, surface run-off has reached the stream system within that period, to a great extent, thus becoming identified with storage. With the exception of such minor losses as channel seepage and surface evaporation, withdrawals of water from the basin, as measured stream flow, constitute the changes in storage. The equation of these factors, including storage, could be stated as follows:

$$\underset{\text{(Daily rainfall)}}{\text{Inflow}} - \underset{\text{(Infiltration)}}{\text{Outflow}} = \text{Change in Storage} + \underset{\text{(Stream flow)}}{\text{Outflow}}$$

It would seem, then, that to preserve the rationality of the unit-hydrograph theory, the distribution of run-off or stream flow throughout the run-off period should be made by applying a distribution graph to rainfall excess, or average daily rainfall minus a depth lost through infiltration.

The pluviagraph is the hydrograph of 100% run-off; but it is not the graph of run-off plus infiltration in that such a definition would imply the distribution of infiltration loss throughout the period of run-off in proportion to run-off. The writer has found the pluviagraph useful as a limiting or enveloping hydrograph from which the hydrograph of a particular flood resulting from a continuous storm period can be reproduced by the application of two related coefficients. First, it must be recognized that the pluviagraph of a unit-time rainfall is nothing more than the mechanical distribution of such rainfall throughout the period of surface run-off in proportions fixed by the distribution graph, establishing the proportionality between the ordinates of the pluviagraph and those of the resulting unit hydrograph. The relation between the unit pluviagraph and the unit hydrograph is linear, therefore,

making the cumulative plot of distributed rainfall (pluviagraph) and stream flow from surface run-off (unit hydrograph) a straight line, the slope of which can be called a storm or flood coefficient. The application of such a coefficient to the pluviagraph of a day's rainfall is the same, in effect, as the distribution of the rainfall excess which remains after the deduction of infiltration from the day's rainfall, and which will produce the unit hydrograph direct.

The relation of the factors involved in the development of the unit graph and the distribution graph is given in Table 11, which presents as an example

TABLE 11.—RELATION BETWEEN FACTORS INVOLVED IN DEVELOPMENT OF UNIT GRAPH AND DISTRIBUTION GRAPH, SUSQUEHANNA RIVER, AT TOWANDA, PA. (AREA, 7 770 SQUARE MILES)

Date: September and October, 1924	Average daily rainfall, in inches	Flow, in Cubic Feet per Second			Distribution graph (percentages)	Pluviagraph, P		Flood coefficient, $C = \frac{Q_s}{P}$
		Observed stream flow, Q_o	Estimated base flow, Q_b	From surface run-off, Q_s		In inches	In cubic feet per second	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
29	....	1 780	1 780	0	0	0	0	.....
30	3.77	45 300	2 500	42 800	14.0	0.527	110 200	0.386
1	....	110 000	3 500	106 500	34.9	1.315	275 000	0.386
2	....	81 200	4 500	76 700	25.1	0.946	197 800	0.386
3	....	51 200	6 000	45 200	14.8	0.557	116 500	0.386
4	....	26 400	7 500	18 900	6.3	0.241	50 200	0.386
5	....	17 100	8 500	8 600	2.9	0.108	22 600	0.386
6	....	13 500	9 000	4 500	1.5	0.057	11 900	0.386
7	....	10 600	9 200	1 400	0.5	0.019	3 970	0.386
8	....	9 060	9 060	0	0	0	0	.....

Date: September and October, 1924	Inflow (Rainfall), I, — Outflow (Infiltration), F = Storage, S, + Outflow (Stream Flow) Q_s						
	In inches	In cubic feet per second	In inches	In cubic feet per second	Cumulative, in cubic feet per second	Change, in cubic feet per second	In cubic feet per second
(1)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
29	....	.....	.....	.....	.....	.....	.....
30	3.77	787 900	2.32	483 300	261 800	+261 800	42 800
1	....	0	....	0	155 300	—106 500	106 500
2	....	0	....	0	78 600	— 76 700	76 700
3	....	0	....	0	33 400	— 45 200	45 200
4	....	0	....	0	14 500	— 18 900	18 900
5	....	0	....	0	5 900	— 8 600	8 600
6	....	0	....	0	1 400	— 4 500	4 500
7	....	0	....	0	0	— 1 400	1 400
8	....	.....	....	.....	.....	.....	.....

the effect of a completely isolated storm of September 29 and 30, 1924, on the Susquehanna River Basin above Towanda, Pa. This storm, although recorded in 2 days, is believed to have been confined to a continuous period approximating 24 hr. The factors of this unit hydrograph are shown in relationship in Fig. 14, together with the cumulative curves of rainfall, infiltration, run-off, and storage.

With less accuracy, but within practical limits, the pluviagraph of a multiple-day storm may be reduced to the approximate flood hydrograph by the

application of a compound linear coefficient, as demonstrated with the typical storm of May 21 to 25, 1919, on the same basin, as reported in Table 12. The cumulative plot of the pluviagraph against that of surface run-off takes a

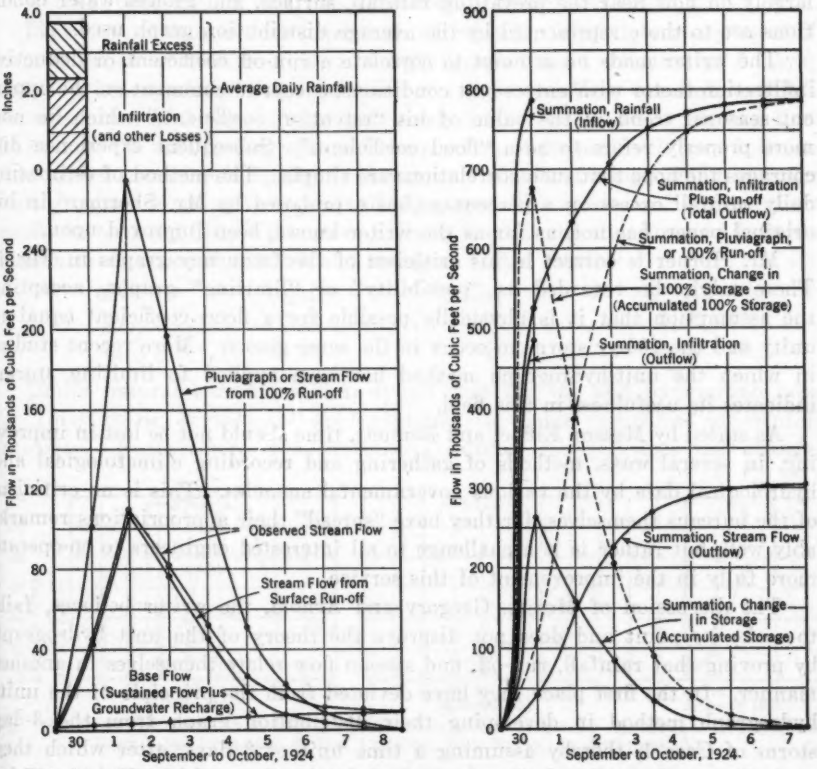

FIG. 14.—RELATIONS OF RAINFALL, INFILTRATION, STORAGE, AND STREAM FLOW, SUSQUEHANNA RIVER, AT TOWANDA, PA.

TABLE 12.—DEVELOPMENT OF FLOOD COEFFICIENTS FOR A FIVE-DAY STORM PERIOD; SUSQUEHANNA RIVER AT TOWANDA, PA.

Date: May, 1919	Average daily rainfall, I, in inches	Average distribution graph (percentages)	Pluviagraph, P*		Flow, in Cubic Feet per Second			Flood coefficient, C	Computed flow, Q_c, in cubic feet per second
			In inches	In cubic feet per second	Observed stream flow, Q_o	Estimated base flow, Q_b	Flow from surface run-off, Q_s		
21.....	0.41	16	0.143	29 900	14 400	6 800	7 600	0.28	8 400
22.....	0.48	37	0.272	26 800	23 900	7 000	16 900	0.28	16 000
23.....	0.56	23	0.386	80 600	54 300	7 000	47 300	0.65	52 400
24.....	0.27	12	0.422	88 300	70 000	7 000	63 000	0.65	57 400
25.....	0.14	6	0.338	70 600	49 000	6 900	42 100	0.65	45 900
.....	....	3	0.225	47 000	37 500	6 800	30 700	0.65	30 600
.....	....	2	0.125	26 100	26 400	6 700	19 700	0.65	17 000
.....	....	1	.....	.....	.....	.....	.....	....	.....

* Includes the distribution of effective antecedent rainfall.

flatter slope for the first two days than that for the remainder of the run-off period, indicating the greater rate of loss through infiltration, which is to be expected. The accuracy with which these results can be obtained will depend largely on how near the prevailing rainfall, surface, and ground-water conditions are to those represented by the average distribution graph used.

The writer made no attempt to correlate a run-off coefficient or deductive infiltration factor with antecedent conditions, except to comment on the apparent seasonal trend in the value of his "retention coefficient," which he now more properly refers to as a "flood coefficient." Subsequent experience discourages the hope that such correlations are simple. The method of estimating daily rainfall excess on a percentage basis, proposed by Mr. Sherman[2] in his original paper, has not, as far as the writer knows, been improved upon.

Mr. Horner is correct in his criticism of the frequency graphs in Fig. 5. These might be regarded as "possibility" or "limiting" graphs, accepting the assumption that it is physically possible for a flood coefficient equal to unity and the 100-yr storm to occur in the same season. More recent studies in which the unit-hydrograph method has been applied to limiting storms indicates its usefulness in this field.

As stated by Messrs. Eiffert and Bennett, time should not be lost in improving, in several ways, methods of gathering and recording climatological and hydrological data by the various governmental agencies. This is no criticism of the bureaus themselves, for they have "spread" their appropriations remarkably well, but rather is it a challenge to all interested engineers to co-operate more fully in the improvement of this service.

The discussion of Messrs. Gregory and Arnold, the writer believes, fails to make its point and does not disprove the theory of the unit hydrograph by proving that rainfall, run-off, and stream flow relate themselves in another manner. In the first place they have deviated from the principle of the unit-hydrograph method in developing their distribution graph from the 3-day storm of Case 1, thereby assuming a time unit of 3 days, after which they apply the resulting distribution graph to 1-day storms. Although the use of a distribution graph developed from their 1-day storm would not have brought the two methods into accord, the absence of any mention of this deviation from the principle of the unit hydrograph would confuse a reader hopelessly who, not being particularly conversant with the unit-hydrograph theory, would attempt to associate the discussion by Messrs. Gregory and Arnold with the writer's paper.

There is only one answer to the discussion by Messrs. Gregory and Arnold. The conclusions, as determined from their hypothetical example, have not been found in Nature. It is possible that their assumptions, among which are uniform rainfall intensity over a 3-day period, no losses to run-off, and no sustained or base flow, may contribute to the abnormal results obtained.

A recent study in which the writer has been engaged, has developed distribution graphs on about fifteen or twenty basins, ranging in area from

[2] "Streamflow from Rainfall by Unit-Graph Method," by LeRoy K. Sherman, M. Am. Soc. C. E., *Engineering News-Record*, April 7, 1932.

500 to 60 000 sq miles. The one nearest in size to the hypothetical problem given by Messrs. Gregory and Arnold (the Red River, at Denison, Tex., of 38 000 sq miles) is selected for the purpose of making the comparison between the natural phenomenon and that visualized by these discussers.

They have presented four conditions of rainfall: Case 1, a heavy rainfall for a duration of 3 days; Case 2, a light rainfall for a duration of 3 days; Case 3, a heavy rainfall for a duration of 1 day; and Case 4, a light rainfall for a duration of 1 day. From their analysis the conclusion is reached that the base of the unit hydrograph and the position of its peak will vary between wide limits, caused principally by the great variation through which average velocity passes in going from low to flood stage. The following list gives their results in terms of "lag" (this term being defined as the time interval between the center of mass of rainfall and the peak of the flood):

Case	Lag-days	Case	Lag-days
1.....	0.53	3.....	4.20
2.....	1.10	4.....	9.00

Under natural conditions it is found that the lag and hydrograph base for 1-day storm periods are approximately the same regardless of rainfall intensity. Furthermore, for continuous rainfall the lag does not change,

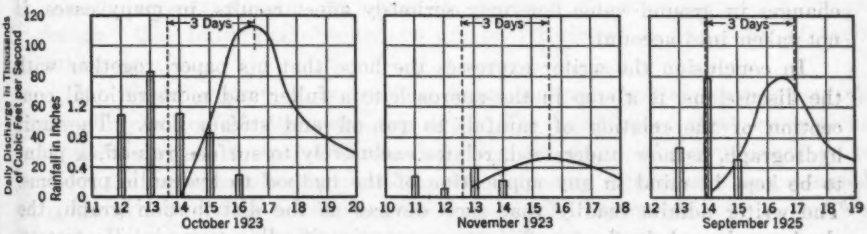

FIG. 15.—LAG OF PEAK RUN-OFF BEHIND CENTER OF MASS OF RAINFALL FOR THREE, FOUR, AND FIVE-DAY STORM PERIODS, RED RIVER, AT DENISON, TEX.

regardless of intensity or deviations from the average. Fig. 15 shows, for the same basin, the lag of 3 days for rainfalls of various intensities for durations of 3, 4, and 5 days.

TABLE 13.—RELATION BETWEEN RAINFALL INTENSITY AND LAG

Date of rainfall	Rainfall, in inches	Date of peak	Lag, in inches	Date of rainfall	Rainfall, in inches	Date of peak	Lag, in inches
(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
December 10 and 11, 1923.....	1.31	December 13..	3	January 4, 1932..	1.50	January 7....	3
March 20 and 21, 1926.....	0.94	March 25.....	4	April 18, 1932...	0.48	April 20 and 21.....	3
March 27 and 28, 1929.....	2.58	March 31.....	3	August 31 and September 1, 1932.....	1.41	September 5..	4
June 14 and 15, 1930.....	0.78	June 18.....	3	.....	.....	.....	.....
December 4 and 5, 1930.....	1.54	December 7...	3	.....	.....	.....	.....

Table 13 shows the relations between rainfall intensity and lag, to the nearest day, for a group of unit hydrographs for the Red River at Denison, Tex.

Comparable results obtained by W. W. Horner, M. Am. Soc. C. E., and F. L. Flynt, Assoc. M. Am. Soc. C. E.,[43] also substantiate the unit-hydrograph theory and disprove the contentions of Messrs Gregory and Arnold.

Mr. Williams directs attention to the use of the unit-hydrograph method as the means of determining the limiting flood on any basin. Such procedure is possible in any case where a record of flow exists sufficiently long to yield the unit hydrographs (preferably not less than three) necessary to the development of the distribution graph for the basin and a sufficient number of well-defined flood periods to indicate the probable limiting value of the flood coefficient. With the distribution graph and the limiting coefficient at hand, outstanding storms that have visited the region may be superimposed upon the basin in a critical position and the resulting flood hydrograph obtained.

The studies of the unit-hydrograph method, referred to in the discussion of Messrs. Hoyt, Harrold, and Snyder, have contributed to the further development and refinement of the method. They have greatly improved the technique of constructing the distribution graph and have shown that seasonal changes in ground-water flow may seriously affect results, in many cases, if not taken into account.

In conclusion the writer expresses the hope that his paper, together with the discussions, is a step in the approach to a fuller and more rational conception of the relation of rainfall to run-off and stream flow. The unit hydrograph, as now understood, relates exclusively to surface run-off, a point to be kept in mind in any application of the method to hydraulic problems. The writer admits readily that such devices as the distribution graph, the pluviagraph, and the flood coefficient are not scientifically precise in all respects. They are tools of practical value, however, the complete explanation of their various functions being bound up in the further development of the unit-hydrograph method.

[43] "Relation between Rainfall and Run-Off from Small Urban Areas," *Proceedings*, Am. Soc. C. E., October, 1934, p. 1135.

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DISCUSSIONS

A GENERALIZED DEFLECTION THEORY FOR SUSPENSION BRIDGES

Discussion

BY D. B. STEINMAN, M. AM. SOC. C. E.

D. B. STEINMAN,[57] M. AM. SOC. C. E. (by letter).[57a]—For the many stimulating and informative discussions that have been contributed, the writer is sincerely grateful. He is glad that his paper has been the means of evoking so valuable a symposium, enriching the science and art of suspension bridge design. The individual discussions are here reviewed in the order of their publication.

Mr. Pavlo presents in convenient form, separate expressions representing the contribution of the tower moments. The addition of these expressions to the respective formulas of the deflection theory for the two-hinged truss yields the more complete formulas for the continuous stiffening truss presented by the writer. This synthetic procedure is only the reverse of the writer's analytical presentation.

The writer's method of obtaining the equivalent or effective moment of inertia of the continuous truss in any span is that of simply multiplying the average chord area by one-half the square of the truss depth. Investigation for several typical designs has indicated, contrary to the opinion of Mr. Pavlo, that this procedure is the most practical method and is sufficiently accurate. Computation of equivalent moments of inertia by comparing unsuspended truss deflections, as recommended by Mr. Pavlo, yields widely discrepant results for some cases of loading; moreover, the results cannot be accepted as correct, since the method does not take into account the effects of the variable suspender loads and of the dead load on the truss deflections.

NOTE.—The paper by D. B. Steinman, M. Am. Soc. C. E., was published in March, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: May, 1934, by E. Pavlo, Esq.; August, 1934, by Messrs. Jonathan Jones, A. Müllenhoff, H. Cecil Booth, Jacob Feld, and Glenn B. Woodruff, Howard C. Wood, and Ralph A. Tudor; September, 1934, by Messrs. L. J. Mensch, A. A. Eremin, Hans H. Bleach, F. H. Frankland, Gustav Lindenthal, Julian W. Shields, A. W. Fischer, and J. M. Frankland; November, 1934, by Messrs. Fredrik Vogt, Leon S. Moisseiff, and A. Mitchell and G. T. Parkin; December, 1934, by Messrs. Sterling Johnston, Harold E. Wessman, and C. H. Gronquist; and January, 1935, by Messrs. W. R. Frederick, Jr., and William H. Yates.

[57] Cons. Engr. (Robinson & Steinman), New York, N. Y.

[57a] Received by the Secretary March 19, 1935.

An accepted scientific measure of rigidity of suspension bridge spans is lacking. The writer agrees with Mr. Jones that change of gradient is a better criterion of comparative rigidity of two designs than mere comparison of maximum or average deflections. The latter procedure, used in the paper for obvious reasons of convenience, familiarity, and expedition, erred on the side of under-statement of the advantages of the continuous type.

The results of the preferred criterion, by comparison of maximum grade changes under live load and temperature, applied to the numerical design examples of this paper, are recorded in Table 9. For the two designs of equal economy, this table shows that the continuous truss is more rigid than the two-hinged type by 40% in the main span and by 10% in the side spans. If only increase in gradient is considered, the rigidity advantages are 6% in the main span and 36% in the side spans. The claim of 6% superior rigidity made in the original paper, therefore, was conservative.

For Design II, representing "equal rigidity" by the method of average deflections, the tabulated values of maximum grade changes show that the continuous truss is more rigid than the two-hinged type by 31% in the main span and by 0% in the side spans. If only increase in gradient is considered, the rigidity is found to be 1% less for the continuous type in the main and 24% greater in the side spans. Hence Design II, which was found to be 5% more economical than the two-hinged design, is also from — 1% to + 31% more rigid in the main span, and from 0% to 24% more rigid in the side spans. Calling it a design of "equal rigidity" erred on the conservative side.

If the average of maximum increase and maximum decrease of gradients for each span is used, the hingeless truss shows 23% superior rigidity in the main span and 22% superior rigidity in the side spans for Design I with equal economy; and Design II, with 5% superior economy, shows 16% superior rigidity in the main span and 11% superior rigidity in the side spans as compared with the two-hinged design. These findings of superior advantages of the continuous type may be summarized in an approximate empirical formula for suspension bridges of 800-ft main span:

$$\text{Gain in Rigidity} + 2\ (\text{Gain in Economy}) = 23\% \quad \ldots\ldots (120)$$

The numerical percentage in this formula, representing combined advantages of the continuous type, will be less for longer spans and greater for bridges of shorter span.

From the viewpoint of ruling gradient, the most significant flexibility characteristic of a suspension bridge is increase of grade in the side spans. Table 9 shows that the superior resistance to increase of gradient in the side spans of the continuous truss, as compared with the two-hinged truss, is marked, amounting to 36% for Design I and to 24% for Design II. The corresponding deflections as computed by Mr. Frederick yield rigidity gains of 38% and 27%, respectively.

Professor Müllenhoff traces the origin of the deflection theory for suspension bridges to Müller-Breslau in a work published in 1881, seven years

TABLE 9.—COMPARISON OF DEFLECTION GRADIENTS

Description	Two-Hinged Trusses	Continuous Trusses	
	Design I	Design I	Design II
Maximum Increase in Grade (Live Load Plus Temperature):			
Main span	2.00%	1.89%	2.02%
Side spans	2.70%	1.98%	2.17%
Maximum Decrease in Grade (Live Load Plus Temperature):			
Main span	2.80%	2.00%	2.13%
Side spans	2.70%	2.44%	2.69%
Grade Change from Temperature Alone:			
Main span	0.49%	0.18%	0.20%
Side spans	0.13%	0.10%	0.11%

before its publication in more developed form by Melan. He inquires as to the possibility of establishing coefficients for reducing the values of H and M given by the elastic theory in order to obtain preliminary design values corresponding to the deflection theory. For M and V in two-hinged stiffening trusses, this has already been done.[58] Similarly, reduction coefficients might be evaluated empirically and recorded for continuous suspension bridges, if such procedure is deemed desirable.

To expedite the application of the deflection theory, tables giving stresses and deflections for spans of various lengths and constants have been developed, but such design tables have not yet been extended to cover continuous and multiple spans.

As a contribution toward facilitating the numerical application of the generalized deflection theory, particularly for continuous suspension bridges, the writer submits Table 10, which gives values of the constants, d, $\frac{L_c}{l}$, h, $\frac{8h}{c^2 l^2}$, al, and $-bl$ for values of cl increasing from 0.1 to 10.0. Most of these constants (with or without the factor, l) occur repeatedly in the working formulas of the paper, and Table 10 expedites their evaluation. The numerical values in Table 10 will also facilitate the setting up of approximate empirical relations.

The function, $\frac{8h}{c^2 l^2}$, in Column (5) Table 10, offers interesting possibilities as a reduction factor for obtaining deflection theory moments and deflections from values given by the elastic theory. Let M_E and M_D denote mid-span bending moments for full-span loading (in any span), computed by the elastic and deflection theories, respectively. Then, upon substitution of the familiar expressions for M_E, Equations (32) yield:

$$M_D = \frac{8h}{c^2 l^2} M_E \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (121)$$

[58] "Suspension Bridges," by D. B. Steinman, M. Am. Soc. C. E., John Wiley & Sons, New York, 1929, pp. 276–282.

Similarly, Equations (33) yield the following convenient formulas for mid-span deflections under full-span loading (in any span):

$$\Delta f = \frac{M_E - M_D}{E\,I\,c^2} \qquad (122a)$$

and,

$$\Delta f = \frac{M_E}{E\,I\,c^2}\left(1 - \frac{8\,h}{c^2\,l^2}\right) \qquad (122b)$$

Equations (121) and (122) are exact for non-continuous spans and closely approximate for continuous spans. The reduction factor, $\frac{8\,h}{c^2\,l^2}$, is conveniently furnished, for varying values of cl, by Table 10. The following approximate empirical relations, deduced from Table 10, may also be convenient:

$$\frac{8\,h}{c^2\,l^2} = 1.1 - h = 0.9 - \frac{L_c}{l} \qquad (123)$$

Hence, Equations (121) and (122) may be written approximately:

$$M_D = (1.1 - h)\,M_E \qquad (124)$$

and,

$$\Delta f = \frac{M_E}{H_w + H}\,(h - 0.1) \qquad (125)$$

Mr. Booth properly emphasizes the need for a specification of maximum allowable deflections for suspension bridges of varying spans and for different classes of traffic. Mr. Johnston makes a valuable pioneer contribution of such a specification for span rigidity. For some time the writer has been engaged on a study of this problem, involving not only the development of a specification based on the flexibility constants of existing satisfactory structures, but also embracing the translation of these rigidity requirements into convenient rules for the quick selection of the best truss depths or moments of inertia for suspension bridges of various types and spans.

Mr. Feld refers to the deviation of the deflected cable curve from the parabola in short spans and the possible effect of such deviation upon the accuracy of the formulas of the deflection theory. Deviations from a parabolic curve are relatively slight in any adequately stiffened suspension bridge, even of short span, and the resulting effect on theoretical stresses is found to be negligible.

Referring to the excellent discussion by Messrs. Woodruff, Wood and Tudor, the writer appreciates their clear and amplifying endorsement of the advantages claimed for the continuous type of suspension bridge as well as their presentation and analysis of the problem of multiple spans.

In their reference to the modified Dana method, which was used in the preliminary phase of the multiple-span studies for the San Francisco-Oakland Bay Bridge, they state that "it has been found by comparison with model

TABLE 10.—VALUES OF CONSTANTS

cl (1)	d (2)	$\frac{L_o}{l}$ (3)	h (4)	$\frac{8h}{c^2l^2}$ (5)	al (6)	$-bl$ (7)	cl (1)	d (2)	$\frac{L_o}{l}$ (3)	h (4)	$\frac{8h}{c^2l^2}$ (5)	al (6)	$-bl$ (7)
0.1.....	0.04996	0.0008	0.00125	1.000	0.00660	0.00340	5.1.....	0.98788	0.61260	0.845	0.2598	8.20076	1.87562
0.2.....	0.09967	0.0033	0.00498	0.996	0.02655	0.01331	5.2.....	0.98903	0.61960	0.852	0.2522	8.40064	1.88528
0.3.....	0.14889	0.0074	0.01114	0.9911	0.05958	0.02976	5.3.....	0.99007	0.62639	0.859	0.2448	8.60053	1.89421
0.4.....	0.19738	0.0131	0.01967	0.9835	0.10550	0.05240	5.4.....	0.99101	0.63296	0.866	0.2376	8.80044	1.90246
0.5.....	0.24492	0.02032	0.03046	0.9747	0.16394	0.08098	5.5.....	0.99186	0.63932	0.873	0.2308	9.00037	1.91009
0.6.....	0.29131	0.02897	0.04337	0.9638	0.23445	0.11513	5.6.....	0.99263	0.64549	0.879	0.2242	9.20030	1.91716
0.7.....	0.33638	0.03891	0.05827	0.9513	0.31645	0.15449	5.7.....	0.99333	0.65146	0.885	0.2178	9.40025	1.92371
0.8.....	0.37995	0.05012	0.07499	0.9374	0.40950	0.19842	5.8.....	0.99396	0.65726	0.890	0.2117	9.60021	1.92973
0.9.....	0.42190	0.06244	0.09336	0.92207	0.51292	0.24650	5.9.....	0.99454	0.66287	0.896	0.2058	9.80018	1.93540
1.0.....	0.46212	0.07576	0.11318	0.90544	0.62606	0.29818	6.0.....	0.99505	0.66832	0.901	0.2001	10.00015	1.94045
1.1.....	0.50052	0.08996	0.13427	0.88774	0.74828	0.35286	6.1.....	0.99552	0.67360	0.905	0.1947	10.20012	1.94522
1.2.....	0.53705	0.10492	0.15645	0.86917	0.87889	0.41003	6.2.....	0.99595	0.67873	0.910	0.1894	10.40010	1.94968
1.3.....	0.57167	0.12051	0.17952	0.84980	1.01721	0.46913	6.3.....	0.99633	0.68370	0.914	0.1843	10.60009	1.95367
1.4.....	0.60437	0.13661	0.20330	0.82980	1.16258	0.52966	6.4.....	0.99668	0.68854	0.919	0.1794	10.80007	1.95743
1.5.....	0.63515	0.15313	0.22761	0.80928	1.31438	0.59108	6.5.....	0.99700	0.69323	0.923	0.1747	11.00006	1.96094
1.6.....	0.66404	0.16995	0.25230	0.78844	1.47195	0.65297	6.6.....	0.99728	0.69779	0.926	0.1701	11.20005	1.96405
1.7.....	0.69107	0.18698	0.27721	0.76736	1.63477	0.71487	6.7.....	0.99754	0.70223	0.930	0.1657	11.40004	1.96700
1.8.....	0.71630	0.20411	0.30221	0.74620	1.80225	0.77643	6.8.....	0.99777	0.70654	0.933	0.1615	11.60004	1.96964
1.9.....	0.73978	0.22128	0.32715	0.72499	1.97391	0.83725	6.9.....	0.99799	0.71073	0.937	0.1574	11.80003	1.97223
2.0.....	0.76159	0.23841	0.35194	0.70388	2.14926	0.89710	7.0.....	0.99818	0.71481	0.940	0.1534	12.00002	1.97450
2.1.....	0.78181	0.25542	0.37648	0.68296	2.32787	0.95573	7.1.....	0.99835	0.71877	0.943	0.1496	12.20002	1.97656
2.2.....	0.80050	0.27227	0.40067	0.66226	2.50938	1.01282	7.2.....	0 99851	0.72264	0.945	0.1459	12.40021	1.97853
2.3.....	0.81775	0.28891	0.42443	0.64186	2.69343	1.06823	7.3.....	0.99865	0.72640	0.948	0.1423	12.60002	1.98028
2.4.....	0.83365	0.30529	0.44771	0.62182	2.87967	1.12185	7.4.....	0.99878	0.73006	0.951	0.1389	12.80001	1.98193
2.5.....	0.84828	0.32138	0.47045	0.60218	3.06784	1.17356	7.5.....	0.99889	0.73363	0.953	0.1355	13.00001	1.98335
2.6.....	0.86172	0.33714	0.49262	0.58298	3.25769	1.22325	7.6.....	0.99900	0.73711	0.955	0.1323	13.20001	1.98479
2.7.....	0.87405	0.35256	0.51416	0.56424	3.44901	1.27087	7.7.....	0.99909	0.74050	0.957	0.1292	13.40000	1.98598
2.8.....	0.88535	0.36761	0.53507	0.54599	3.64157	1.31639	7.8.....	0.99918	0.74380	0.959	0.1262	13.60000	1.98720
2.9.....	0.89569	0.38228	0.55532	0.52825	3.83523	1.35977	7.9.....	0.99926	0.74702	0.961	0.1233	13.80000	1.98830
3.0.....	0.90515	0.39657	0.57491	0.51103	4.02982	1.40108	8.0.....	0.99933	0.75017	0.963	0.1204	14.00000	1.98928
3.1.....	0.91379	0.41046	0.59381	0.49433	4.22521	1.44029	8.1.....	0.99939	0.75324	0.965	0.1177	14.20000	1.99012
3.2.....	0.92167	0.42396	0.61203	0.47815	4.42130	1.47738	8.2.....	0.99945	0.75623	0.967	0.1150	14.40000	1.99098
3.3.....	0.92886	0.43705	0.62957	0.46249	4.61798	1.51250	8.3.....	0.99950	0.75916	0.968	0.1125	14.60000	1.99170
3.4.....	0.93541	0.44976	0.64644	0.44736	4.81516	1.54562	8.4.....	0.99955	0.76201	0.970	0.1100	14.80000	1.99244
3.5.....	0.94138	0.46207	0.66265	0.43275	5.01278	1.57688	8.5.....	0.99959	0.76480	0.971	0.1076	15.00000	1.99304
3.6.....	0.94681	0.47399	0.67821	0.41865	5.21076	1.60628	8.6.....	0.99963	0.76753	0.973	0.1052	15.20000	1.99364
3.7.....	0.95175	0.48554	0.69313	0.40504	5.40906	1.63390	8.7.....	0.99967	0.77019	0.974	0.1030	15.40000	1.99426
3.8.....	0.95624	0.49672	0.70742	0.39192	5.60761	1.65981	8.8.....	0.99970	0.77280	0.975	0.1008	15.60000	1.99472
3.9.....	0.96032	0.50753	0.72110	0.37928	5.80640	1.68410	8.9.....	0.99973	0.77534	0.977	0.0987	15.80000	1.99520
4.0.....	0.96403	0.51798	0.73421	0.36711	6.00537	1.70687	9.0.....	0.99975	0.77783	0.978	0.0966	16.00000	1.99550
4.1.....	0.96740	0.52810	0.7467	0.3554	6.20450	1.72818	9.1.....	0.99978	0.78027	0.979	0.0946	16.20000	1.99600
4.2.....	0.97045	0.53788	0.7587	0.3441	6.40378	1.74800	9.2.....	0.99980	0.78265	0.980	0.0926	16.40000	1.99632
4.3.....	0.97323	0.54733	0.7702	0.3332	6.60317	1.76661	9.3.....	0.99982	0.78492	0.981	0.0907	16.60000	1.99666
4.4.....	0.97574	0.55648	0.7811	0.3228	6.80266	1.78386	9.4.....	0.99983	0.78727	0.982	0.0889	16.80000	1.99680
4.5.....	0.97803	0.56532	0.7915	0.3127	7.00223	1.80005	9.5.....	0.99985	0.78951	0.983	0.0871	17.00000	1.99715
4.6.....	0.98010	0.57387	0.8015	0.3030	7.20186	1.81506	9.6.....	0.99986	0.79170	0.983	0.0854	17.20000	1.99732
4.7.....	0.98197	0.58214	0.8109	0.2937	7.40156	1.82896	9.7.....	0.99988	0.79384	0.984	0.0837	17.40000	1.99768
4.8.....	0.98367	0.59014	0.8200	0.2847	7.60131	1.84193	9.8.....	0.99989	0.79594	0.985	0.0821	17.60000	1.99784
4.9.....	0.98522	0.59787	0.8287	0.2761	7.80109	1.85407	9.9.....	0.99990	0.79800	0.986	0.0805	17.80000	1.99802
5.0.....	0.98661	0.60536	0.8369	0.2678	8.00091	1.86519	10.0.....	0.99991	0.80002	0.987	0.0789	18.00000	1.99820

tests to give more accurate values for deflections than those obtained by the 'deflection theory.' The error in the latter theory is doubtless due largely to the neglect of the change in span lengths." This defect may have been true of the deflection theory as previously available; but in the formulas of the generalized deflection theory as presented in the paper for multiple-span suspension bridges, full consideration is given to the changes in span lengths.

In the final design formulas presented by the discussers, Equation (64) for the main cable is, as they state, identical with the writer's Equation (47) with the terms transposed; and Equation (63) for the tie cable can be shown to be effectively identical with Equation (46), which the writer has reduced from the more exact mathematical relationship to more compact, workable form with certain simplifying approximations, after having verified their numerical permissibility.

After setting up these two essential formulas corresponding to Equations (46) and (47) of the paper, the discussers present an attractive semi-graphical method for recording and applying the resulting numerical values. In either the graphic or analytical procedures, a method of "cut and try" or successive approximation, must be used in satisfying the governing relationship of Equation (48*a*). The graphic method has the advantage of facilitating visual inspection and interpolation of results in the various intermediate steps. The writer presented the analysis in the form of compact summation equations for the sake of maximum directness and maximum generality of application, covering not only the simplest case, but also the conditions of tower resistance and truss continuity.

Mr. Mensch cites precedents in other fields of structural analysis where the problems have been simplified so that differential equations are not required for their solution, and he suggests a possibility of similar simplification of the suspension bridge theory. Opinions may differ, however, on the importance of this objective. In the derivations of the paper, the application of differential equations is negligible, and the resulting working formulas have been reduced to a form requiring not even a knowledge of the calculus for their application. In the solution of such problems of structural analysis, the first essential step is, with the use of suitable mathematical tools, to establish correct equations for the unknowns sought; then comes the simplification of these formulas to compact, workable form for practical application; and, finally, there follows the development of tables or charts computed from the formulas, for expediting design.

The types of structural problems cited by Mr. Mensch bear little or no similarity to the suspension bridge problem. Any opportunity for further material simplification of the derivation and working formulas of the deflection theory is not apparent.

Mr. Eremin contributes generalized formulas for the loading terms, A, B_1, B_2, G_1, and G_2, applicable to a variety of loading cases. These are of interest, although they would generally be less convenient in practical application than the reduced formulas given in the paper for the specific loading cases. The demonstration by Mr. Eremin that the effect of suspender elongation may be neglected, is a matter of useful record.

Dr. Bleich contributes an independent confirmation of the writer's results and conclusions, particularly in reference to the claims of superior efficiency for the continuous type of suspension bridge, and he extends the indication of superior advantage to a range of greater span lengths than was anticipated by the writer. Dr. Bleich correctly points out that the comparison of quarter-point deflections in the main span is a better criterion of comparative rigidity of two designs than a comparison of mid-span deflections. On this basis he finds, for a span-length of 1 575 ft, that the continuous design is 17% more rigid than the two-hinged design of the same economy; the comparison of side span deflections in the same study shows 20% advantage in rigidity for the continuous design. For two designs of equal depth and rigidity at this span length, Dr. Bleich finds for the continuous design an economic advantage of 13% by comparison of moment graphs and 33% by comparison of assumed moments of inertia. This apparent economic advantage, however, vanishes if the two designs under comparison are adjusted to different truss depths to balance the relative values of I and M. This explains the anomalous showing of greater economic advantage at a span length of 1 575 ft than was found by the writer at a span length of 800 ft. The writer does not advocate continuous construction for highway suspension bridges exceeding 1 000-ft main span, but a longer span may be the economic limit for continuity in suspension structures designed for railway use.

Various possibilities, hitherto unrealized, of advantage and economy for bridges of the suspension type are pointed out by Mr. F. H. Frankland. In the short-span and multiple-span field, suspension construction should find increased application. In this connection it may be noted that, as pointed out by Mr. Lindenthal, there are conditions under which multiple-span suspension bridges can be used economically and effectively without the provision of tie cables.

Mr. Lindenthal contributes an interesting and valued discussion relating to the development of suspension bridge theory and practice. He emphasizes the fact that the minor approximations made in the derivation of the formulas of the paper are completely justified, since the stiffening truss of a suspension bridge is only a secondary member, on which the safety of the structure is not dependent. He states: "Nothing serious can happen to a suspension bridge as long as the anchorage, towers, and cables are sound." This important and governing fact should be kept in mind in discussing any further refinements in stiffening truss analysis.

Confirming and extending the writer's conclusions, Mr. Shields presents the results of the application of the generalized deflection theory to a suspension design of 400-ft span. The superior economy of the continuous type over the two-hinged truss, indicated by the writer as 5% for an 800-ft span, is found by Mr. Shields to increase to 12% for the shorter span length of 400 ft. The finding of increased economy for shorter spans was predicted and, as mentioned also by Messrs. Pavlo and F. H. Frankland, demonstrates that the continuous type of stiffening truss with the improved analytical theory will increase the importance of the suspension bridge in the short-span field.

If the comparisons were made on the basis of equal deflection gradients instead of merely equating mid-span deflections, the hingeless type would show still greater economic advantage. The discussion of Dr. Bleich covering the comparative results obtained for a suspension design of 1 575-ft main span, combined with the report of Mr. Shields on results for a 400-ft span, together serve to extend the range and generalize the practical application and conclusions of the theory as given by the writer.

As was anticipated, Mr. Shields finds that the shortness of span introduces some modifications of the general loading conditions outlined in the paper.

The use of hyperbolic functions instead of exponential functions in the formulas of the deflection theory, as advocated by Mr. Fischer, has previously appeared in the work of other writers. The basis of a choice between the two forms will generally be that of personal preference. The writer has found, by experience and comparison, that the exponential form with the introduction of the d-function (which may be considered either a hyperbolic or an exponential function) yields maximum compactness of formulas and facility of analysis. A table of exponential functions will facilitate substitution in those sections of the working formulas where the general d-function has not already simplified their form and application.

In an attempt to offer a simplified procedure, Mr. Fischer applies the familiar principle that the point of maximum bending moment for a fixed loading of length, k, is defined by writing the expression for shear equal to zero. In the notation of the paper, using Equation (10), this value of x for a given value of k is simply given by the formula,

$$e^{2cx} = \frac{C_2}{C^1} \qquad \text{.....(126)}$$

which should be compared for brevity with Mr. Fischer's Equation (82).

Unfortunately, however, this proposed short-cut, eliminating the use of trial load lengths, does not yield the results sought. The problem is not to find the maximum bending moment producible by a given loading, but to find the loading producing the maximum bending moment at a given point of the span and then to find the resulting value of this maximum bending moment.

In general, the maximum bending moment thus obtained for a given section will be greater than the more restricted value yielded by Mr. Fischer's proposed procedure; the two values will agree only at or near the points of absolute maximum bending moments in the span such as those accidentally selected by Mr. Fischer for his numerical illustration. If a bending moment curve is drawn for a given load length, k, some point on this moment curve other than the maximum may be a governing design value; the envelope of such moment curves for various values of k will be the curve of maximum bending moments used in design. Instead of this comprehensive and lengthy procedure, the use of a few trial load lengths for each section, as illustrated in the paper, is a convenient practical expedient, yielding the desired results.

The interpolation formula (Equation (83)) contributed by Mr. J. M. Frankland for use in connection with Table 1, or Table 10, for values of d

should prove useful when an adequate table of hyperbolic functions is not available. Ordinarily, linear interpolation between the tabulated values of cl should suffice, and closer interpolation should be necessary only for the lower values of d in Table 1, or in Table 10.

The independent derivation by kinematical analysis of Equation (100) identical with Equation (22) is a welcome additional check of this equation which underlies the H-formulas of the paper. It is re-assuring to know that this important equation is accurate despite the several assumptions involved in its simplified derivation as a work equation.

Mr. Frankland's suggested correction (Equation (102)) of the temperature term in Equation (22) is also welcome. This correction, writing L = total distance between anchorages, in place of L_t, in all formulas for suspension bridges, although negligibly affecting the results, should be accepted as an improvement. It has the advantage of greater simplicity, while correcting the H-formulas for the previously neglected effect of temperature change in the steel towers and in the suspenders. The correctness of this modification may be verified, easily and independently, by adding temperature terms for towers and suspenders in the formula for the internal work, W_2, preceding Equation (22); substitution and reduction then yields L instead of L_t in Equation (22) and all subsequent formulas.

A slight additional correction in the temperature term can be written to take into account differences of elevation of steel tower bases and anchorage ends of cables, but such further correction is negligible.

Professor Vogt refers to the influence line solution proposed by the late Hans H. Rode, M. Am. Soc. C. E. In this method of obtaining the load lengths for maximum moments or shears, the values of influence ordinates for M or V at a given point for unit concentrations moving along the span are computed for two assumed values of c^2, corresponding to maximum and minimum probable values of $(H_w + H)$, in order to account for the lack of direct proportion between load and stress.

On the assumption that there will not be an excessive spread between the influence lines thus plotted for the two values of c^2, the approximate load length for maximum moment or shear can be scaled, the value of H for this loading can be found by interpolation from the two influence lines for H, and the final load length and moment or shear can then likewise be found by interpolation between the two influence lines for M or V. If the final values of H and M or V are computed by the formulas for distributed loading, it should be necessary to obtain the influence line ordinates only near the critical points of zero moment or shear.

This method of Professor Rode is not, strictly speaking, an influence line solution; but, if it is not too lengthy or inaccurate, it may have an advantage, in some instances, over the generally more practical solution by trial. At any rate, this suggested influence-line procedure holds some possibilities for further development. The values of H and M or V for the unit loads would be computed from the general formulas of the deflection theory for concentrated loads.

Mr. Moisseiff questions whether including the effect of suspender elongation and tower shortening will not serve to increase the tower moments of the continuous stiffening truss as analyzed by the formulas of the paper. As stated elsewhere, the effect upon H of temperature changes in the lengths of towers and suspenders is included in the analysis by merely substituting for L_t in the formulas the simpler quantity, L, which is the sum of the horizontal cable spans. Due to the great sectional area of the towers as compared with the cross-section of the cables, the shortening of the towers under live load is relatively negligible.

In the continuous type, at and near the towers, the effect of live load elongation of suspenders is of interest only for that case of loading which produces, simultaneously, maximum negative tower moment and maximum positive shear at the tower. For this loading condition, with the near side span fully loaded and the main span only partly loaded (for a minor fraction of its length out from the tower), the value of H in the cable is small; in the numerical example of the paper, it amounts to only about one-sixth of the maximum value of H. Consequently, the suspender forces, and therefore the changes in suspender lengths, will also be relatively small. The generalized deflection theory shows, moreover, that for this case of loading the continuous truss, under the action of live load and temperature change, relieves the cable slightly of dead load near the tower, so that several suspenders adjacent to the tower actually shorten rather than elongate. This would tend to decrease the negative tower moment. Actual computation for the effect of suspender elongation on the stiffening truss of Design II indicates a decrease in both the negative moment and the shear at the tower, which decrease amounts, however, to only a fraction of 1 per cent. It thus appears conclusive that the effect of change of length of suspenders on the analysis of the continuous stiffening truss may be neglected safely. This conclusion is confirmed independently by analysis in the discussion by Mr. Eremin.

Near the tower, the continuous truss does not follow the deflections of the cable; consequently, the truss is not "unduly" strained as surmised by Mr. Moisseiff. Near the tower, the continuous truss deflects less than the cable; hence the suspender shortening. The theory assumes that the truss follows the cable deflections and, therefore, yields near the tower more severe stresses than actually occur. The difference, however, is negligible.

In the remainder of the span, where positive bending moments govern, the suspenders elongate and the truss deflects slightly more than the cable. Suspender elongation tends to reduce positive bending moments, so that here, also, the theory yields more severe stresses than actually occur. Here, again, the difference is negligible (a small fraction of 1 per cent). In addition to being negligible in effect, the assumptions that have been challenged are on the safe side.

Messrs. Mitchell and Parkin present formulas for the G-functions which they have derived for the case of a span divided into three loading segments. Completely general formulas for the G-functions, as mentioned in the paper,

may be written as follows for any segment, m, of any span having any number of loading segments with any uniform loads (or zero loading) in any segments:

Let $p_1, p_2, p_3 \ldots =$ the uniform load in the first, second, third, etc., segments from the left end of the span, up to p_m in Segment m (note $p_0 = 0$); $k_1, k_2, k_3 \ldots =$ the clear distance of the first, second, third, etc., segments from the left end of the span, up to k_m for Segment m (note $k_1 = 0$); $q_1, q_2, q_3 \ldots =$ the uniform load in the first, second, third, etc., segments from the right end of the span, up to $q_m = p_m$, in Segment m (note $q_0 = 0$); and, $z_1, z_2, z_3 \ldots =$ the clear distance of the first, second, third, etc., segments from the right end of the span, up to z_m for Segment m (note $z_1 = 0$, and $k_m + m + z_m = l$). Then, in any segment (m) of any span,

$$G_1 = -\frac{(1-d)}{4d}\Sigma_L\left[(p_n - p_{n-1})\,(e^{ck} + e^{-ck})\right] + \frac{1+d}{4d}\Sigma_R\left[(q_n - q_{n-1})\,(e^{cz} + e^{-cz})\right] \quad \ldots\ldots(127a)$$

$$G_2 = \frac{1+d}{4d}\Sigma_L\left[(p_n - p_{n-1})\,(e^{ck} + e^{-ck})\right] - \frac{1-d}{4d}\Sigma_R\left[(q_n - q_{n-1})\,(e^{cz} + e^{-cz})\right] \quad \ldots\ldots(127b)$$

and,

$$G_3 = \tfrac{1}{2}\Sigma_L\left[(p_n - p_{n-1})\,(e^{ck} + e^{-ck})\right] \quad \ldots\ldots(127c)$$

in which Σ_L denotes the summation for all the segments from the left end of the span up to and including the segment, m, and Σ_R denotes the similar summation for all the segments from the right end of the span up to and including the segment, m.

Applying the general rule expressed by Equations (127), all the G-formulas for the different loading cases tabulated in Article 9 may be written by inspection. G-formulas for other loading cases or for other span segments may likewise be written directly from Equation (127).

Mr. Johnston makes a pioneer contribution of a specification for rigidity of suspension bridges. He presents a chart (Fig. 13) which gives his recommendations for allowable change of gradient for highway suspension bridges of varying lengths of spans and numbers of traffic lanes. For a 4-lane structure of 800-ft main span, the chart yields an allowable grade change of 2.22%, which should be compared with the computed values recorded in Table 9.

Change of gradient in a span is generally significant only when the change is such as to increase the normal (unloaded) grade of the roadway. If Fig. 13 were to be interpreted as for allowable increase of grade and the allowable base percentage of 2.00% were to be increased to 2.25 or 2.50%, the chart would better fit established and proved practice and would constitute a practical specification for suspension bridge rigidity. It should be noted, in this connection, that the distinction between increase and change of grade is ordinarily of consequence only in the continuous type.

A working specification of allowable flexibility of a highway suspension bridge should also be governed by the location of the bridge, since a grade change which would be considered excessive in a large city or for heavy trucking traffic might be altogether proper in a less heavily traveled site. For computing the maximum change of grade in any span, a particular design loading as to length and placement in each span must be assumed. If it appears that such a theoretical loading is seldom likely to occur, that fact should be borne in mind in choosing the value of the maximum permissible grade change.

The recommendation of Mr. Johnston that the total augmented grade of the roadway should never exceed 5% is likewise subject to qualification. Higher ruling gradients are demanded and approved in some localities, and appear to be justified by the improvements that have been made in automotive equipment. Urban traffic conditions which might set the maximum augmented grade of a span at 5% would not obtain in rural districts or in regions where steeper grades are the rule.

Professor Wessman calls attention to the omission of the shear effect in writing the differential equation for the truss deflection. It is the writer's opinion that the relative shallowness of the stiffening truss fully justifies the neglect of shear contribution to the deflection, even in the case of the continuous type. This view is substantiated by the results obtained in the computations for effective moments of inertia of the truss; in these computations, the contribution made by the diagonals to the total deflections is found to be negligibly small in comparison with that made by the chords.

The formulas for concentrated loads, contributed by Mr. Gronquist, will be found useful in computing deflections under heavy concentrations, and in applying the partial influence-line solution of Professor Rode as suggested by Professor Vogt.

In the thoughtful discussion contributed by Mr. Frederick, he supports the thesis that angular deflection or grade change is a better index of comparative rigidity than mere comparison of maximum deflections. In his studies of comparative economy, he uses the refinement of varying the truss depths, even as between main span and side spans of the same design. For the brief numerical comparisons at the end of the paper the writer hesitated to go to this extreme, although it would have facilitated the presentation of "balanced" designs.

Of course, investigators using different methods of evaluating relative economy and rigidity will obtain varying numerical results. Although quantitative findings vary, the qualitative conclusions are in accord: All agree on the important point that continuous suspension design for span lengths of the range under consideration are more rigid or more economical (possibly both) than the two-hinged type.

The maximum angular deflections reported by Mr. Frederick in Table 6 are in sufficiently close agreement with the side-span grade increases recorded by the writer in Table 9, but the economic comparisons in Table 7 do not appear entirely consistent. Nevertheless, the final results of Mr. Frederick's

analysis serve only to support and strengthen the writer's claim of superior advantage for the continuous type. In comparison with the two-hinged design, Mr. Frederick's results show 25% gain in economy plus 38% gain in rigidity for Continuous Design I; more than 1% gain in economy plus 27% gain in rigidity for Continuous Design II; and 6% gain in economy plus 18% gain in rigidity for Continuous Design III. All these combinations exceed the measure of superior advantage for this span length recorded by the writer in summary form in Equation (120), and the conclusion recorded in Equation (120), in turn, exceeds the more conservative claims made in the original paper.

Mr. Yates properly refers to the Rondout Bridge, at Kingston, N. Y., as a pioneer among modern suspension structures, from the viewpoints of continuity, short span, concrete deck, and design for modern heavy highway loading. Had it been available in 1920, the generalized deflection theory would have facilitated and improved the stress analysis and proportioning of sections for that bridge, and would have yielded a more efficient design.

Conclusions.—In conclusion, it may be stated that the objectives of the paper have been confirmed and amplified by the discussions. The deflection theory for suspension bridge analysis has been generalized and extended to cover structures of any number of spans, continuous or non-continuous, symmetrical or unsymmetrical, and with or without tie cables. Theory and formulas have been reduced to a new degree of compactness and simplicity, and procedures have been developed to facilitate and expedite application. The advantages of continuous stiffening trusses, in economy and rigidity, have been confirmed. Correct analysis as a prerequisite to the development of possibilities in the multiple-span field of suspension bridge design has been presented. Increased possibilities for the suspension type in the short-span bridge field have been suggested. Criteria for comparing and specifying the rigidity of suspension structures have been introduced. The reliability, economy, and workability of the deflection theory have been confirmed, and its range of application extended, so as to relegate the comparatively wasteful and unscientific elastic theory to obsolescence.

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DISCUSSIONS

THE RESERVOIR AS A FLOOD-CONTROL STRUCTURE

Discussion

BY MESSRS. GERARD H. MATTHES, AND GEORGE R. CLEMENS

GERARD H. MATTHES,[12] M. AM. SOC. C. E. (by letter).[12a]—Reservoirs possess important potentialities as flood-control structures, as ably brought out by the author, but the significant fact remains that surprisingly few have been built for that purpose despite the urgent need for flood control that exists on many streams. In 1920 there were in the United States a paltry dozen reservoirs and retarding basins devoted exclusively to the control of floods. This number includes such minor units as the retarding basin on Dry River, near Watervliet, N. Y., and another, even smaller, on Paxton Creek, for the protection of Harrisburg, Pa. From estimates compiled by the writer in 1920 there were then in the world at large fewer than 100 flood-control structures of these two types. More than one-half of these were in Europe, 21 being in one river basin in Germany. Until a few years ago this paucity of flood-control reservoirs had not changed materially. In the United States, during the past six years, many reservoir projects for flood control have been surveyed and reported on, but to date (1935) few of these projects have reached the construction stage. The chief obstacle appears to be their unwarranted cost.

In general, a storage basin for the handling of large volumes of water ranks among the most costly of hydraulic structures, whatever may be its purpose. In hydro-electric development it is common for the dam and reservoir to represent the largest single item of investment cost, and the one upon which most often hinges the economic feasibility of the entire development. Experience teaches that reservoirs designed exclusively for flood control have been found less likely to be warranted on strict economic considerations, than have reservoirs designed for any other single purpose. There is a steadily growing tendency to build multiple-use reservoirs such

NOTE.—The paper by George R. Clemens, Assoc. M. Am. Soc. C. E., was published in May, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1934, by C. Stanley Maxwell, Jun. Am. Soc. C. E.; and December, 1934, by Messrs. F. Knapp, and Charles S. Bennett.

[12] Prin. Engr., Office of Pres., Mississippi River Comm., Vicksburg, Miss.

[12a] Received by the Secretary March 11, 1935.

that the cost may be prorated among two or more uses, often wholly unrelated. Flood control, which is often included in such combinations, is gaining steadily in importance and is emerging from the incidental by-product class in which it has largely figured in the past, into a well-defined reliable status. Boulder Dam, on the Colorado River, which is designed primarily for flood-control and at which power will be generated, as well as water furnished for irrigation, domestic supply, and navigation purposes, is the outstanding example of the multiple-use type. Norris Dam, on the Clinch River, Tygart River Dam, on the head-waters of the Ohio River, and Fort Peck Dam on the Missouri River, are to serve not less than two purposes each. The flood-control reservoirs being built for the Los Angeles District will be operated so as to make available the impounded waters for irrigation and water supply. The reservoirs of the Muskingum Conservancy District will serve other purposes beside flood protection to the Muskingum Valley. Credit should be given the larger of the existing Western irrigation reservoirs which derive their water supply mainly from flood run-off, for having reduced flood flow more or less effectively although they have never been operated for that purpose.

The conclusion is inescapable that the multiple-use reservoir is destined to play an increasingly important rôle in the future. This is supported by the following considerations: (*a*) In the United States, to date, only densely populated, highly industrialized areas can afford the cost of reservoirs for the sole purpose of flood control; (*b*) costly impounding structures for the sole purpose of preventing flood damage to small towns and farm lands situated in natural flood-plains can rarely be justified on a strictly economic basis; (*c*) to waste flood waters immediately they have been impounded, as must be done at retarding basins or at reservoirs designed solely for flood control, is contrary to the modern concept of water resources utilization; and, (*d*) multiple use of reservoirs divides the burden of cost between two or more interests, and the increased benefits which result are more likely to place the undertaking on a sound economic basis.

Many of those who have written about the incompatibility of operating a reservoir, or system of reservoirs, for joint power development and flood control have overlooked the fact that this incompatibility resides not so much in the physical aspects of operation as in the divergence of interests which exists between corporate use and public conservation of water. The former is ever concerned with earning adequate returns to the investor; the latter with the greatest good to the public at large. There can be no unanimity of purpose, nor any "give and take," as between such divergent interests.

The author has done a great service in analyzing the various possibilities of the combined uses of reservoirs and by pointing out the respective shortcomings and merits of the various combinations. It should be emphasized, however, that even in the most advantageous physical set-ups outlined by him, absolute agreement between the interests sharing the storage space in a reservoir is essential. This agreement should amount to a unity of pur-

pose and should take the form of unified control, if the best results are to be obtained because, in any combination, one usage may be called upon, at times, to sacrifice to the other. Conceivably, this may be unavoidable even where separate storage space is allotted to each usage. Unified control by a single interest, whether Federal, State, or other competent public agency, appears to be the only satisfactory solution whenever flood control is involved.

The practical operating difficulties of flood-control reservoirs have been much under-rated. This is attributable, in large measure, to the excellent performance of the flood-control system of the Miami Conservancy District, Ohio, which has caused a widespread but erroneous belief that the automatic retarding basin principle utilized there is applicable to any drainage basin where the topography is at all favorable toward retarding-basin construction. Primarily, it is adapted to relatively small water-sheds where flood run-off exhibits little variation in synchronization of flood crests from the various tributaries, and permits the development of a simple plan of operation. Once installed, this plan admits of no variation because the outlets are of fixed size. The operation of the retarding basins in the Miami Valley is singularly favored by the almost ideal crowfoot arrangement of the main tributaries. An important consideration which entered into the adoption of retarding basins there was the desire of the people of the Valley to preclude any possible use of the dams by power or other interests, whereby any of the flood-control features might later be sacrificed.

The retarding principle cannot be relied upon for satisfactory performance in large drainage basins, or in any flood-control system in which a large number of dams and reservoirs are required, or where the tributary system is at all complex. Automatic control of one tributary in such cases is likely to lead to unfavorable synchronization with flood crests from other tributaries similarly controlled, or not controlled, or with flood crests in the parent stream, as pointed out by the author. Other drawbacks of the retarding-basin principle are: Inflexibility in emergencies; steadily decreasing rate of outflow as the water level above the dam sinks; inability to flush out the river channel below for the purpose of removing sediment deposits; and impracticability of raising the height of the dam and the water level at some future time when loss of storage capacity due to sedimentation should make such a step desirable, without having to rebuild the fixed outlet to meet the increased head imposed on it.

As against these limitations gate control possesses many advantages, not the least of which is that a gate-controlled dam can be made to operate as a retarding dam, whenever this may be advantageous, by merely locking the gates at a predetermined height so as to pass water at the same rates of discharge as a dam provided with a retarding outlet. At other times such a dam permits of gate operation, and this, during the flood season and especially during emergencies, becomes of inestimable value. Gate control means virtually complete control. While gates can never be made as fool-proof as a fixed retarding outlet, and require replacement at intervals, it is entirely practicable to design them so as to prevent their being tampered with, short of the use of explosives.

Whatever may be the hazards of gate control with its implied reliance on human judgment and ingenuity, duly commented on by the author, the fact remains that there are many localities in the United States where no other type of reservoir control can safely be used. For instance, wherever major floods are caused by tropical storms the possible recurrence of which at 7-day or 8-day intervals must always be anticipated, reservoirs for flood control must be designed so as to permit being emptied completely on short notice in order to be available for the coming flood. Nothing short of complete control by means of ample gate capacity in the hands of experts can meet such a situation. At such times it will be better to cause the rapid release of the impounded waters to overtop the river banks, thereby causing moderate inundation, than to permit the crest of the succeeding flood to go over the spillway and cause a disastrous inundation. This is choosing the lesser of two evils. A retarding basin, in such instance, would admit of no choice and might become a liability rather than an asset. The 7-day to 8-day interval between great floods is a matter of record on a number of streams along the South Atlantic Coast, and is equally likely to occur on the rivers of the Gulf Coast. Of course, the ideal plan would give reservoirs or retarding basins storage capacities sufficient to absorb the flood waters from more than one tropical storm, but this is seldom feasible for economic reasons.

Gate control is indispensable also in areas where cloudburst run-off must be coped with, especially where the flood-control reservoir controls a small drainage basin. Unless its operation is in competent hands, it may well happen that cloudburst run-off immediately following a major storm finds the reservoir full to the spillway level and must pass over the spillway. This cloudburst problem is peculiar to all streams in the Rocky Mountain region, to a large number of Appalachian streams, and to most of the streams in the arid regions. The retarding basins in the Miami Valley were not designed with either of these considerations in mind. The region is out of the path of tropical storms and cloudburst floods are not of great intensity there. The largest, at Englewood, if filled to near spillway level, requires 23 days to empty; two, similarly, would be emptied in from $7\frac{1}{2}$ to 8 days, and the remaining two in $4\frac{1}{2}$ days.

The flood-control system of the Miami Conservancy District is the most successful of its kind in the United States, but it is not a useful yardstick to apply indiscriminately to other flood problems. It would not be surprising if this system should be destined to remain in a class by itself. It never would have been economically feasible had it not been for the high valuation of property, great density of population, and the importance of the industries located in the flood zone. The 36 000 acres within the basins furnish protection to 80 000 acres of valley lands, a ratio of 1 acre of basin lands to about 2 acres protected, some of the farm land acreage being only partly protected. In addition to the five dams, cut-offs, channel dredging, and levees were required, bringing the net construction cost to nearly $30 000 000, no portion of which is covered by Federal grant or assistance.

The entire investment cost is being repaid by the interests benefited, about one-half by direct assessment on 77 000 pieces of property, and the remainder through tax levies for protection to public property of cities and counties. The 67 000 acres of farm lands collectively were assessed less than 5% of the total direct assessment levied. The remaining 13 000 acres of city and industrial property pay the other 95 per cent. As the farm lands are by no means second rate, these figures are significant in illustrating how little farm land can afford to be assessed on the strict benefit basis required under the Conservancy Act. It proves also that the remainder of the property in the flood zone was worthy of flood protection in perhaps an exceptional degree, as it shouldered successfully a heavy burden. There are few localities in the United States in need of flood protection to-day that could afford to finance a purely flood-control project on any such scale.

GEORGE R. CLEMENS,[13] ASSOC. M. AM. SOC. C. E. (by letter).[13a]—A number of reservoir projects which involve flood control, either as a sole purpose or as an important one of several purposes, have been undertaken since this paper was prepared. Many other projects are being actively considered, and some will probably reach the construction stage in the near future. It has been extremely interesting to note that general consideration of the various types of projects fits readily into the classification outlined in the paper. Particularly is this true of the analysis of combined projects and the possibilities and limitations of combined use of the same reservoir for several purposes. The discussions of Mr. Knapp and Mr. Matthes have touched on combined use. Certain illustration of the original ideas presented may be helpful.

It was pointed out that, in general, combined use might be divided into two classes: (*A*) Combined purpose storage (combined use for more than one purpose of the same storage space); and (*B*) combined interval storage (a combination of several purposes by allotting portions of the reservoir to each separate use).

It was stated (and the discussion supports this viewpoint) that Use (*A*) is more difficult to obtain than Use (*B*). An example of each is presented for illustration.

On the Sacandaga River above Albany, N. Y., the Hudson River Regulating District has constructed a reservoir for the purpose of "regulating the flow of the Hudson and Sacandaga Rivers as required by the public welfare." This involves control of floods; increase of low-water flow in part of the New York State canals; increase of low-water flow in the interest of public health; and regulation of flow for water power. These several more or less conflicting requirements are met by use of water stored in a reservoir of 690 000 acre-ft capacity, operated by the District to comply as nearly as possible with the requirements of the law that created it. There is a large measure of power development of the 750 ft of fall in the Hudson and

[13] Senior Engr., Mississippi River Comm., Vicksburg, Miss.

[13a] Received by the Secretary March 11, 1935.

Sacandaga Rivers from tide-water at Troy, N. Y., to the storage reservoir dam 80 miles up stream, as shown on the profile (Fig. 14). Obviously, there are times when each interest must sacrifice something to the other. According to data furnished by Edward H. Sargent, M. Am. Soc. C. E., Chief Engineer of the District, the results given in Table 9, were obtained in the

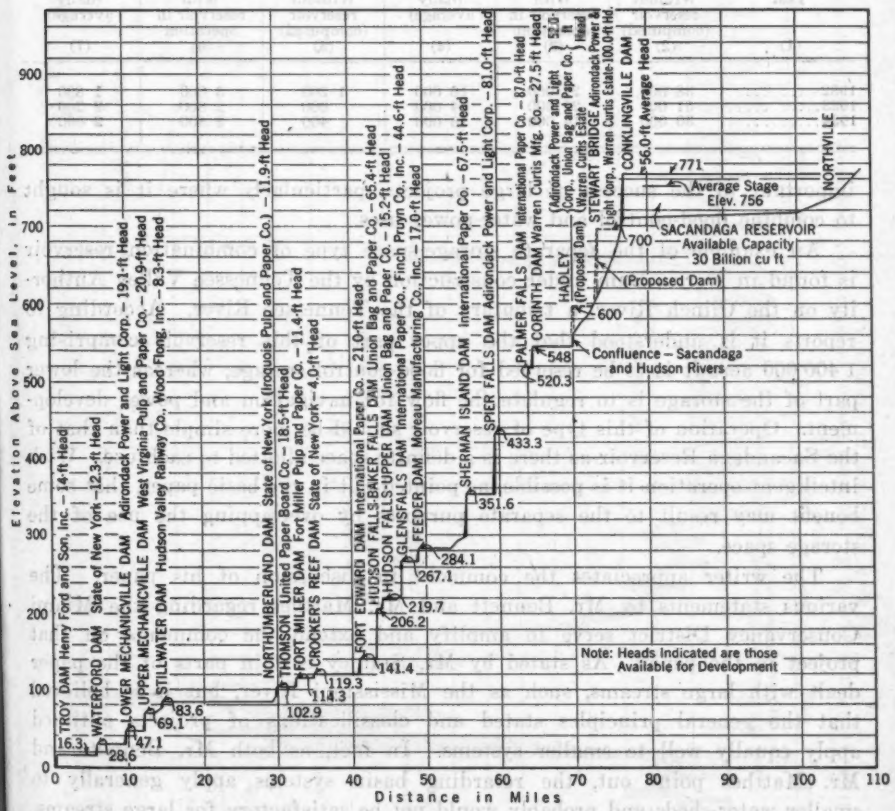
Elevation Above Sea Level, in Feet
Distance in Miles
TROY DAM Henry Ford and Son, Inc. – 14-ft Head
WATERFORD DAM State of New York – 12.3-ft Head
LOWER MECHANICVILLE DAM Adirondack Power and Light Corp. – 19.1-ft Head
UPPER MECHANICVILLE DAM West Virginia Pulp and Paper Co. – 20.9-ft Head
STILLWATER DAM Hudson Valley Railway Co., Wood Flong, Inc. – 8.3-ft Head
NORTHUMBERLAND DAM State of New York (Iroquois Pulp and Paper Co.) – 11.9-ft Head
THOMSON United Paper Board Co. – 18.5-ft Head
FORT MILLER DAM Fort Miller Pulp and Paper Co. – 11.4-ft Head
CROCKER'S REEF DAM State of New York – 4.0-ft Head
FORT EDWARD DAM International Paper Co. – 21.0-ft Head
HUDSON FALLS-BAKER FALLS DAM Union Bag and Paper Co. – 65.4-ft Head
HUDSON FALLS-UPPER DAM Union Bag and Paper Co. – 15.2-ft Head
GLENSFALLS DAM International Paper Co., Finch Pruyn Co., Inc. – 44.6-ft Head
FEEDER DAM Moreau Manufacturing Co., Inc. – 17.0-ft Head
SHERMAN ISLAND DAM International Paper Co. – 67.5-ft Head
SPIER FALLS DAM Adirondack Power and Light Corp. – 81.0-ft Head
PALMER FALLS DAM International Paper Co. – 87.0-ft Head
CORINTH DAM Warren Curtis Mfg. Co. – 27.5-ft Head
HADLEY (Adirondack Power and Light Corp., Union Bag and Paper Co.) 52.0-ft Head (Proposed Dam) (Warren Curtis Estate
STEWART BRIDGE Adirondack Power & Light Corp., Warren Curtis Estate-100.0-ft Hd.
CONKLINGVILLE DAM
56.0-ft Average Head
NORTHVILLE
771
Average Stage Elev. 756
SACANDAGA RESERVOIR
Available Capacity 30 Billion cu ft
700
(Proposed Dam)
600
548
Confluence – Sacandaga and Hudson Rivers
520.3
433.3
351.6
284.1
267.1
219.7
206.2
141.4
119.3
114.3
102.9
83.6
69.1
47.1
28.6
16.3
Note: Heads Indicated are those Available for Development
0
100
200
300
400
500
600
700
800
900
0
10
20
30
40
50
60
70
80
90
100
110

FIG. 14.—PROFILE OF THE HUDSON AND SACANDAGA RIVERS FROM TROY TO NORTHVILLE, N. Y.

years 1932, 1933, and 1934. The effect of operation during the year 1933, as reflected in the flows of the Hudson River at Spier Falls, N. Y., is shown in Fig. 15.

Detail study of the operation charts reveals many interesting facts, but the most important is that, with a unified control, it is possible to obtain a joint use of storage in spite of the many conflicting demands. As can be noted from Table 9, floods are materially reduced and low-water flow is substantially increased. It is probable that the most important factor in obtaining the results with this combination reservoir is the unified control. This point was raised in Mr. Matthes' discussion and is believed to be very

TABLE 9.—EFFECT OF OPERATING SACANDAGA RESERVOIR ON HUDSON RIVER AT SPIER FALLS, NEW YORK

(Units are in cubic feet per second)

Year	Maximum Flow		Decrease (daily average)	Approximate Low		Increase (daily average)
	Without reservoir (computed)	With reservoir in operation		Without reservoir (computed)	With reservoir in operation	
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1932.....	38 000	22 000	16 000	1 200	3 000	1 800
1933.....	51 000	27 000	24 000	600	2 800	2 200
1934.....	30 000	14 000	16 000	400	2 400	2 000

important to the success of large projects, particularly where it is sought to combine flood-control and water-power uses.

An example of the separate storage-space type of combination reservoir is found in Norris Dam under construction by the Tennessee Valley Authority on the Clinch River, a tributary of the Tennessee River. According to reports it is understood that the upper part of this reservoir comprising 1 400 000 acre-ft is to be reserved for flood-control storage, whereas the lower part of the storage is to regulate the flow for navigation and power development. Operation of this type of reservoir should be more simple than that of the Sacandaga Reservoir as there is a definite space allotted to each use. With intelligent operation it is possible, as pointed out in the basic paper, that some benefit may result to the separate purposes by overlapping the use of the storage space.

The writer appreciates the comment in discussion of his paper. The various statements by Mr. Bennett and Mr. Matthes regarding the Miami Conservancy District serve to amplify and extend the comments on that project in the paper. As stated by Mr. Stanley certain parts of the paper dealt with large streams, such as the Mississippi River, but it is believed that the general principles stated and classifications of projects outlined apply equally well to smaller systems. In fact, as both Mr. Bennett and Mr. Matthes point out, the retarding basin systems apply generally to smaller water-sheds and probably would not be satisfactory for large streams, such as the Mississippi. Control of outlets is quite generally a desirable feature even if the main purpose is to retard flood flow.

As brought out in Mr. Matthes' discussion the use of reservoirs for flood control alone has been limited It is also rather seldom that the cost of construction can be borne by the interests directly benefited. There are many indeterminate benefits however, to a reservoir system that at present are impossible to evaluate and assess. The general effects on business and transportation of the elimination of flood damage or even a flood hazard in any considerable area is something difficult to set down in dollars and cents. Employment of men on constructive work during periods of general unemployment has certain value. Prevention of interruption of business opera-

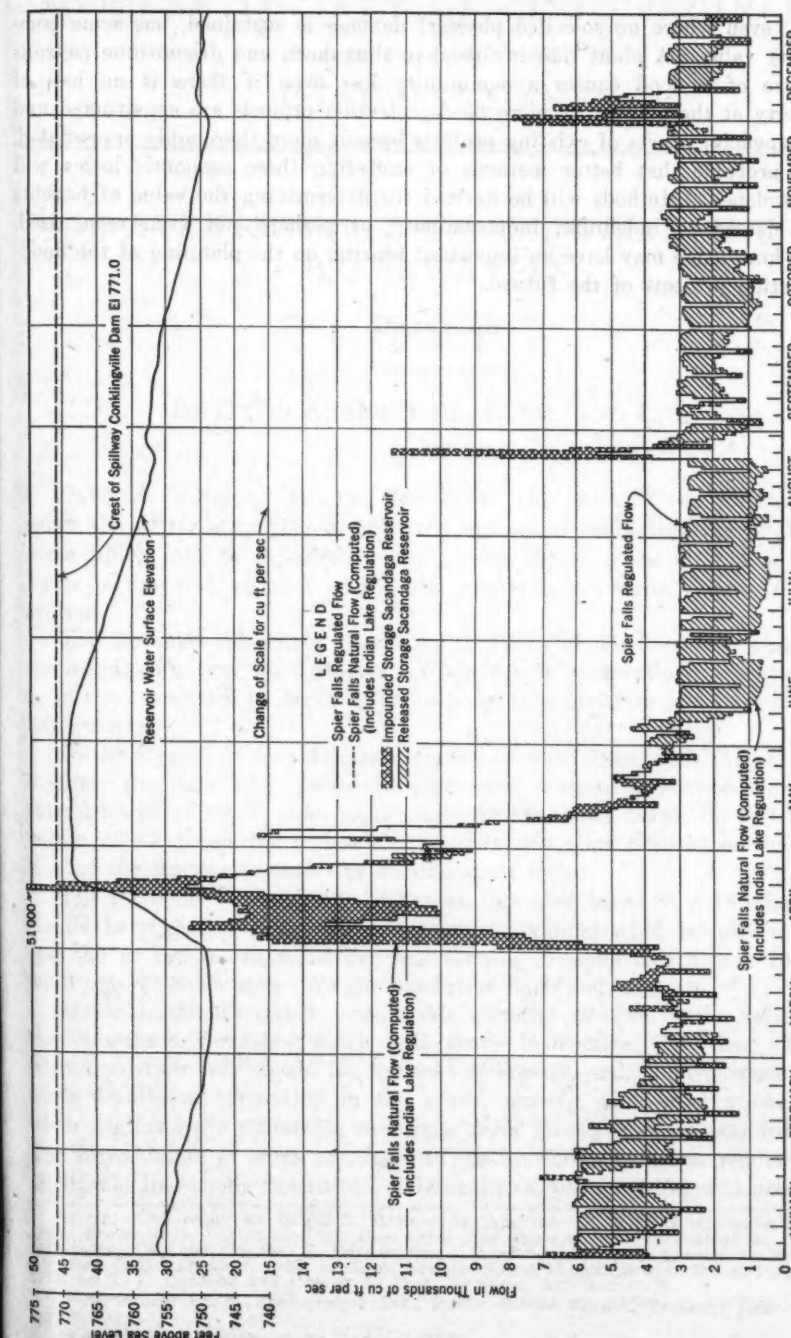

FIG. 15.—HYDROGRAPH OF HUDSON RIVER AT SPIER FALLS, N. Y., SHOWING EFFECT OF SACANDAGA RESERVOIR REGULATION FOR 1933 AND WATER SURFACE ELEVATION IN RESERVOIR.

tions, even where no so-called physical damage is sustained, has some community value. A plant that is forced to shut down and discontinue payrolls because of a flood causes a community loss even if there is no loss of property at the plant. As more flood-protection projects are constructed and the beneficial effects of existing projects become more thoroughly appreciated, it is probable that better methods of analyzing these economic losses will be developed. Methods will be devised for determining the value of benefits now classed as indefinite, indeterminate, or perhaps not even recognized. Such knowledge may have an important bearing on the planning of the flood-protection projects of the future.

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DISCUSSIONS

WAVE PRESSURES ON SEA-WALLS AND BREAKWATERS

Discussion

BY DAVID A. MOLITOR, M. AM. SOC. C. E.

DAVID A. MOLITOR,[21] M. AM. SOC. C. E. (by letter).[21a]—In offering this paper the writer attempted to outline a method of evaluating the total wave force which may be expected in any locality where a sea wall or a breakwater is proposed, so that a safe and relatively economic design could be evolved.

The formulas for height of a wave in terms of wind velocity and fetch are admittedly very approximate. When direct observations are available, as in the cases cited by Mr. Fowler, recourse to speculative estimates becomes unnecessary.

In the absence of reliable measurements of wave dimensions for any given locality, the data here presented give very acceptable answers for most inland bodies of water, more particularly for the Great Lakes, for any known values of wind velocity and fetch; actually, the time element or the duration of the storm also enters as an important factor.

The reason for revising the Stevenson formulas, based on a 78-mile wind, should be apparent, as in many cases the maximum wind velocity reported may be as low as 40 miles per hr, whereas in some localities a 100-mile wind may be encountered for the particular fetch and exposure.

Major Leeds has added considerable material relative to the validity of the Stevenson formulas which will prove instructive. He has supplied what the writer left unsaid for the sake of brevity, and, finally, comes to the same conclusions presented in the paper, namely, that observational data when available are preferable to results from grossly approximate formulas. The introduction of wind velocity into the Stevenson formulas renders them applicable to varying conditions. Naturally, a 40-mile wind will produce a

NOTE.—The paper by David A. Molitor, M. Am. Soc. C. E., was published in May, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: September, 1934, by Charles E. Fowler, Esq.; December, 1934, by Charles T. Leeds, M. Am. Soc. C. E.; January, 1935, by A. A. Eremin, Assoc. M. Am. Soc. C. E.; and February, 1935, by T. L. Condron and Chester L. Post, Members, Am. Soc. C. E.

[21] Structural Engr., Procurement Div., Public Works Branch, Treasury Dept., Washington, D. C.

[21a] Received by the Secretary March 2, 1935.

lesser wave height than a 100-mile wind, whereas the Stevenson formulas apply only to a 78-mile wind. There are many cases on the Great Lakes in which only a 40-mile wind has occurred in the special direction affecting a given breakwater.

Reference is made to the value of k in Equation (12) which it is stated may have a possible maximum value of 2.00. This value might be utilized for wind velocities of about 100 miles per hr, and for storms of long duration. Smaller values of 1.3 to 1.7 are perfectly safe for waves produced by wind velocities from 30 to 70 miles, respectively, lasting not more than 24 hr.

The writer agrees with the preference expressed by Major Leeds for rubble-mound breakwater construction whenever such type is suitable. A factor of safety of, say, two against sliding and overturning, for crib designs, is likewise considered adequate for maximum storm conditions of record. The cost of infrequent repairs resulting from damage due to abnormal storms is also regarded as more economical over long periods of time.

The use of Equation (19) should be restricted to waves striking in rather deep water. Whenever deep-water waves approach a beach, the shoaling causes the waves to veer materially. Finally, they break on the beach along lines nearly parallel to the shore, in which case the formula would give values that are too small.

Mr. Eremin presents a curious method attributed to Sainflou, which he applies to Magann's Pier and finds a rather close agreement with the writer's analysis. However, this must be regarded as quite accidental and should not be accepted as a criterion for other cases.

The discussion contributed by Messrs. Condron and Post brings out some interesting points which deserve special attention. Their analysis of the Calumet Harbor Breakwater, serves as a splendid illustration of the tendency to over-estimate rather than under-estimate the wave force.

Although there is no question regarding the stability of this structure as designed, the writer wishes to record the results of an analysis such as he considers in accordance with the method advocated in the paper.

The 55-mile wind from the northeast agrees with the records of the U. S. Weather Bureau as given in Fig. 5. However, a fetch of 300 miles, which is the entire length of Lake Michigan, presupposes a storm of wider extent and longer duration in this direction than has ever occurred. It is writer's judgment that a fetch of 150 miles from a northeast direction would better fit this case, with the wave approaching normally to the breakwater, instead of at an angle. Furthermore, as the depth of water in front of the structure is 32 ft, the deep-water wave will not undergo any shoaling effect before striking.

It is quite inadmissible to apply wave pressures observed at Duluth Harbor directly to the Calumet Harbor Breakwater without due allowance for the different exposures. Each case must be treated in strict accordance with the special conditions governing the same. Finally, it must be clear that, in order to build up full wave pressure, the resisting surface must be of sufficient height (about 17.5 ft at Calumet Harbor) while the top of the

pier is only 6.5 ft above still-water level. Therefore, this wave encounters only a relatively small obstruction, and does not build up pressures as great as those estimated by Messrs. Condron and Post.

The analysis herein presented is believed to approximate the true conditions more accurately, and will illustrate the correct application of the writer's method to the steel pile structure at Calumet Harbor as shown in Figs. 10 and 11.

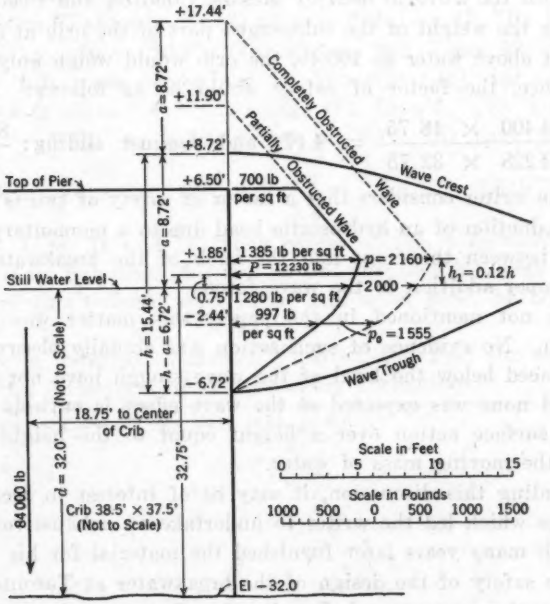

FIG. 11.—CALUMET HARBOR BREAKWATER, LAKE MICHIGAN.

Hence, for a maximum wind velocity of 55 miles per hr and a normal fetch of 150 miles, the following results are obtained: By Equation (1), $h = 0.17 \sqrt{55 \times 150} = 15.44$ ft; by Equation (3), with $\frac{L}{h} = 15.3$ for $V = 55$, gives $L = 15.3\ h = 236$ ft; by Equation (4*a*) for $d > 1.84\ h$, $a = \frac{h}{2} + \frac{h^2}{L} = 8.72$ ft and $h - a = 6.72$ ft. By Equation (7) for $\frac{d_o}{L} = \frac{d}{L} + 2\left(\frac{h}{L}\right)^2 = 0.144$ (for which Table 3 gives $c = 0.858$, $\beta = 1.358$, and $\mu = 1.165$), find $v = 2.26\ c \sqrt{L} = 29.67$ ft per sec.

Furthermore, Equation (14) gives $v_o = 7.11\ \mu \frac{h}{\sqrt{L}} = 8.3$ ft per sec. Equation (15) then gives $p = 1.5\ (v + v_o)^2 = 2\,160$ lb, and $p_o = 0.72p = 1\,555$ lb per sq ft; also, $h_1 = 0.12\ h = 1.85$ ft.

The foregoing calculations make it possible to plot the pressure curve for the completely obstructed wave as shown in Fig. 11, from which the pressure curve for the partly obstructed wave is deduced.

The total effective pressure, $P = 12\,230$ lb per lin ft of structure, is obtained by fixing the height of the pressure area at Elevation 11.9, which is a mean level between the top of the pier and the full wave height at Elevation 17.44. This gives a maximum pressure of 1 385 lb per sq ft, as compared with the 2 370 lb used by Messrs. Condron and Post.

Assuming the weight of the submerged part of the crib at 50 lb per cu ft and the part above water at 100 lb, the crib would weigh only 84 400 lb per lin ft. Hence, the factor of safety would be as follows: Against overturning: $\frac{84\,400 \times 18.75}{12\,228 \times 32.75} = 4.17$; and, against sliding: $\frac{84\,400 \times 0.5}{12\,228} = 3.45$. The writer considers that a factor of safety of two is sufficient.

The introduction of an hydrostatic head due to a momentary difference in water level between the front and the back of the breakwater is not considered a proper addition to the wave force.

Although not mentioned in the paper, this matter was given careful consideration. No evidence of such action was actually observable as dynamometers placed below the level of the wave trough have not indicated any pressure, and none was expected as the wave effect is entirely dynamic and confined to surface action over a height equal to the height of the wave and due to the moving mass of water.

In concluding this discussion, it may be of interest to recall briefly the circumstances which led the writer to undertake an exhaustive study of this subject which many years later furnished the material for his paper.

When the safety of the design of the breakwater at Toronto Harbor was challenged by the contractor, before its actual construction was undertaken in 1915, it became necessary to establish proof of adequate design. In the absence of any authenticated method of procedure, it was necessary to devise a method of appraising approximately, the probable wave force for the particular exposure and depth of water, and then to show by comparison that the same method of analysis, when applied to numerous existing structures, gave results which agreed well with their actual behavior after they had weathered many severe storms. It was thus a case of first inventing a method of analysis and then of proving the method to be reliable.

After the claim of inadequate design was refuted by the writer's method of approach, a thorough investigation was made by a Board of Consulting Engineers, with the result that the structures were built without increased dimensions. They have now (1935) weathered all storms for the past nineteen years without showing any signs of weakness.

The writer wishes to express his thanks and appreciation to those who have contributed to this discussion. The points brought out will serve to emphasize certain matters on which divergence of opinions may be expected.

The following corrections will be made in the paper [21b] when it is published in *Transactions*: (1) Following Equation (7), page 658, substitute "in which, $c = \sqrt{\frac{b_s}{a_s}}$ when $\frac{b_s}{a_s}$ expresses the ratio of the semi-minor to the semi-major axes of the surface orbits. Two other factors, β and μ, required in Equations (13) to (16), together with values of c, are given in Table 3"; (2) change headings of Columns (2) and (3), Table 3, to read, respectively, "Values of c" and "Values of β"; (3) on page 662, Line 13, "By Equation (1) $h = 0.17\sqrt{35 \times 260} = 16.5$"; (4) on page 662, Lines 22 and 23, "The crest of this 16-ft wave determined by Equation (4*b*), for shallow water waves, was 10.8 ft above * * *"; (5) page 663, Line 5, "(By Equation (15), $p = 1.4\ (31 + 9.57)^2 = 2\ 300$ lb per sq ft)"; (6) page 665, Line 14, "* * * and by Equation (15) (with $\frac{kw}{2g} = 1.3$), $p = 1\,556$ lb per sq ft"; (7) page 666, Lines 18 and 19, "The height of this deep-water wave above still-water level by Equation (4*a*) is $a = 8.65$ ft"; and (8), page 670, Line 26, "The height of this reduced wave above still-water level, by Equation (4*b*), will be, * * *."

The writer is indebted to Messrs. Condron and Post for correspondence calling attention to several of the preceding corrections.

[21b] *Proceedings*, Am. Soc. C. E., May, 1934, p. 653.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

AN ASYMMETRIC PROBABILITY FUNCTION

Discussion

BY ARNE FISHER, ESQ.

ARNE FISHER,[31] ESQ. (by letter).[31a]—In several places in this paper Professor Slade uses such terms as "homograde series," "homograde frequency functions," and "partly bounded functions"; and in Part IV he discusses a certain function which he terms "the most general frequency function" which, in a sense, would seem to imply that this type of function includes all other known types, such as the Gaussian or Laplacean exponential error curve, the Poisson exponential, Gram's series, and the error curves introduced by Kapteyn and Bachelier. It should be noted that the author does not state what he understands by the word, "homograde." The technical terms, "homograde" and "heterograde," were first introduced and defined in the theory of statistics by Charlier and have a perfectly definite meaning.[32] The term, "homograde," as used by Professor Slade apparently stands for something quite different and his so-called "homograde functions" appear to be certain frequency functions defined by Equation (45) or Equation (11) of his paper. These equations are nothing but special cases of the frequency function:

$$y = \frac{1}{n\sqrt{2\pi}} e^{-0.5t^2} \qquad (52)$$

in which, $t = \left[\frac{h(x) - m}{n}\right]$, which in the case of $h(x) = x$ reduces to the well-known law of error by Laplace and Gauss.

In 1856, the Danish Minister of Finance, C. G. Andrea, demonstrated[33] that although the spread of "hits" of rifle bullets, when measured in reference to a vertical or horizontal diameter, follow the well-known exponential error

NOTE.—The paper by J. J. Slade, Esq., was published in October, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: January, 1935, by Messrs. Gordon P. Williams, and H. Alden Foster; February, 1935, by Messrs. R. D. Goodrich, and F. T. Mavis; and March, 1935, by L. Standish Hall, Assoc. M. Am. Soc. C. E.

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[31a] Received by the Secretary January 24, 1935.

[32] "Mathematical Theory of Probabilities", by Arne Fisher, p. 128, Macmillan Co., New York, 1930.

[33] "Bestemmelsen af Skudsikkerheden ved Skydning mod verticale Skiver", by C. G. Andrae, 1856.

law, it will be found, on the other hand, that the spread of the hits when measured from the center of the target will be characterized by the following skew frequency curve:

$$P(x) = \frac{2x}{a^2} e^{-0.5x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

where $P(x)$ is the probability that a bullet will strike a target at a distance, x, from the center. Furthermore, if $P_1(x)$ is the normal frequency or probability curve,

$$P_1(x) = \frac{1}{n\sqrt{2\pi}} e^{-0.5x^2 : n^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (54)$$

the probability curve for, say, x^2, will differ from Equation (54)[34].

In general, therefore, if a certain statistical variate, x, is normally distributed (that is, if it is subject to the Laplacean probability curve, or the error law of Gauss), a function of x, say $H(x)$, will not necessarily be distributed normally. On the other hand, it is also true that if the variate, x, is not normally distributed, a function of x, say $h(x)$, may be so distributed. Therefore, it should be possible, at least in theory, to throw a skew or non-normal curve into normal form through a proper transformation of the variate. x.

As a matter of fact, in 1889[35], Thiele, a Danish astronomer and actuary, discussed the general problem of determining the frequency curves, not only for x but for various functions of x. He mentions that, if the distribution of the variate, x, as actually observed, is found to be skew, or to deviate from the normal probability curve, x may be considered as a "fictitious" observation of the "pseudo-normal" variate, $t = h(x)$, which latter function is supposed to follow the Gaussian error law; that is, if $t = h(x)$ is distributed according to a known normal error law of exponential form, it is possible to find the frequency distribution of the inverse function, $x = g(t)$.

A brief outline of some of the more elementary parts of Thiele's work (based on a communication to the writer from the late Dr. J. P. Gram in 1903) is, as follows:

1.—Arrange the observed frequencies, $F(x)$, in class intervals of the variate, x.

2.—Accumulate the values, $F(x)$ (that is, in the form $\sum_{-\infty}^{x} F(x)$), successively, or step by step, from lower to upper limits of class intervals.

3.—Calculate the accumulated relative frequencies, or $\sum_{-\infty}^{x} \phi(x) = \dfrac{\sum_{-\infty}^{x} F(x)}{N}$

in which, $\sum_{-\infty}^{\infty} F(x) = N$ equals the size of the observed sample; that is, the total number of observed individuals.

[34] "The Mathematical Theory of Probabilities," by Arne Fisher, 1930, p. 236.

[35] "Almindelig Iagttagelseslaere" ("A General Doctrine of Observations"), by T. N. Thiele, 1889.

4.—The curve under Step (3) will be a monotonely increasing function, gradually rising from zero to an upper limit which never can be greater than 1.

5.—It is evident, therefore, that a unique, or a one-to-one, correspondence exists between $\sum_{-\infty}^{x} \phi(x)$ and the probability integral, $I(t) = \int_{-\infty}^{t} e^{-0.5t^2}\, dt$, so that to every value of x in $\sum_{-\infty}^{x} \phi\ (x)$ there will be a corresponding value of t in $I(t)$.

6.—Plot on co-ordinate paper the values of t corresponding to x. The form of the graph or curve drawn through these points will give a general idea of the function of x that will be normally distributed, although x itself may not be normally distributed.

7.—The functional expression of $t = h(x)$ may assume a great variety of forms, some of which are:

$$t = ax + b = (x - M) : \sigma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55)$$

which is the usual form of t in the exponential curve of error;

$$t = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56)$$

special forms of which are ax^2, ax^3, etc.; and, $t = \log x$, or,

$$t = [\log (x - a) - m] : n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (57)$$

8.—When it is known that there is both a lower finite limit,, x_0, and an upper finite limit, x_n, the following form is very useful:

$$t = h\,(x) = \left[\log \left(\frac{x - x_0}{x_n - x}\right) - m\right] : n \quad \ldots\ldots\ldots\ldots\ldots\ldots (58)$$

A correlation coefficient, for instance, has a lower limit of -1 and an upper limit of $+1$. A probability can never be less than zero or greater than 1. At any one instant, the busy selectors over a telephone exchange can never be less than zero or greater than the number of subscribers. (In Poisson's formula there can be theoretically an infinite number of calls.)

Among these various transformations the writer first proposes to consider Equation (57) because it is the one that he has discussed somewhat in detail elsewhere and the one which Professor Slade has criticized. Under the heading, "Transformations of the Variate," the author cites Equation (10) as the definition of a new generic function credited to the writer. This statement is in error; no where has the writer made use of the function as claimed in Professor Slade's Equation (10). The generator that the writer used[10] was of the form:

$$\phi_0\,(x) = \frac{N}{n\sqrt{2\pi}}\, e^{-0.5[h(x)]^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (59)$$

in which, $h(x) = \left[\dfrac{\log x - m}{n}\right]$.

[10] "The Mathematical Theory of Probabilities", by Arne Fisher, 1930, p. 236 *et seq.*

Moreover, the writer has emphasized the fact that the logarithmic transformation in question, presupposes a lower limit, a, located at a distance of u units from the mean value of the variate (or first-order semi-invariant, $\lambda_1(x)$, of x) and defined by the relation, $\lambda_1(x) - u = a$. The value of u is the real root in the cubic equation,[36]

$$\lambda_3 u^3 - 3\lambda_2^2 u^2 - \lambda_2^3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (60)$$

in which, λ_2 and λ_3 denote the semi-invariants of second and third order.

The logarithmically transformed generic function, as the writer uses it, becomes, therefore, Equation (59), with $h(x) = \left[\frac{\log(x-a) - m}{n}\right]$, and not the function that Professor Slade attributes to the writer in his Equation (10).

This feature is important in connection with the author's "partly bounded function," which he defines by Equation (11). In introducing this allegedly "new" function Professor Slade writes,

"Notice, in the first place that Equation (11) is similar to the logarithmically transformed function, Equation (10), differing from it only in the power, c^2, to which it is raised. The difference is fundamental, however. It is just this added parameter, c, that makes Equation (11) more than a 'normal' for the logarithms of the variate and which brings about a remarkable simplification of the formulas. It is this c, indeed, that makes Equation (11) a true generalization of Equation (1)."

In this case the author has magnified the mathematical symbol, c, out of all proportion; his Equation (11) is identical to the logarithmically transformed generator in Equation (59) which he has criticized in his discussion of some of the writer's work.

As proposed by Professor Slade in Equation (11) his function is of the form:

$$y = A\, e^{-[g(x)]^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (61)$$

in which, $g(x) = c\,[\log(x + b) + \log d]$; whereas the generator, as the writer has used it, is of the form of Equation (59), with $h(x) = \left[\frac{\log(x-a) - m}{n}\right]$, which simply states that the quantity, $\log(x - a)$, is normally distributed around a mean value, m, with a dispersion (or standard deviation) equal to n.

This generator is algebraically identical to the "partly bounded function" in the paper by Professor Slade; in fact, $A = N\,(n\sqrt{2\pi})^{-1}$; $c^2 = 1 : 2n^2$; $\log d = -m$; and, $b = -a$ (or u). The author's criticism of the type of generator in Equation (59), therefore, also applies to his own "partly bounded function" (Equation (11)).

By way of illustration it may be of interest to apply the methods advocated by the writer to Professor Slade's "Example *a*." The author uses moments, whereas the writer has preferred to use semi-invariants, which, as

[36] "The Mathematical Theory of Probabilities," by Arne Fisher, 1930, p. 239.

a rule, are easier to handle in numerical calculations. However, when the moments are taken around the mean as origin, it is found that the first, second, and third moments, μ_1, μ_2, and μ_3, are equal to the semi-invariants, λ_1, λ_2, and λ_3; therefore, $\mu_1 = \lambda_1 = 2.335$; $\mu_2 = \lambda_2 = 1.442$; and $\mu_3 = \lambda_3 = 3.607$.

The equation for determining the value of u is, therefore: $3.607u^3 - 3\ (1.422)^2\ u^2 - 1.442^2 = 0$; or,

$$3.607\ u^3 - 6.238\ u^2 - 2.998 = 0 \qquad (62)$$

for which the real root is found to be $u = 1.948$ (with $u^2 = 3.795$). Consequently, the lower limit, a, equals 2.335 − 1.948, or 0.387.

Moreover,[36] $n^2 = \log\ (1 + 1.442{:}3.795) = \log 1.381 = 0.323$; $n = 0.5683$; $m = \log\ (1.948) - 1.5\ (0.323) = 0.6668 - 0.4845 = 0.1823$; and, $N = 251{:}e^{0.1823 + 0.1615} = 251{:}1.416 = 177.93$; so that finally:

$$\phi_0\ (x) = \frac{177.93}{0.5683\ \sqrt{2\ \pi}}\ e^{-0.5[h(x)]^2} = \frac{313.1}{\sqrt{2\ \pi}}\ e^{-0.5[h(x)]^2} \qquad (63)$$

in which, $h(x) = \left[\dfrac{\log\ (x - 0.387) - 0.1823}{0.5683}\right]$.

The computation of the ordinates is given in Table 12. It will be noticed that the final results are exactly the same as those in Table 4 and are reached

TABLE 12.—Computations for Determining Ordinates of Frequency Curve
(See Table 4)

x (1)	log $(x-0.387)$ (2)	(2)−0.182 (3)	(3)=t : 0.568 (4)	$\phi_0\ (t)$ (5)	313.1(5) (6)
1	−0.489	−0.671	−1.181	0.1986	62.2
2	0.478	0.296	0.521	0.3483	109.1
3	0.960	0.778	1.370	0.1561	48.9
4	1.285	1.103	1.942	0.0605	18.9
5	1.529	1.347	1.371	0.0240	7.5
6	1.725	1.543	2.717	0.0099	3.1
7	1.889	1.707	3,005	0.0044	1.4
8	2.030	1.848	3.254	0.0020	0.6
9	2.153	1.971	3.470	0.0010	0.3
10	2.263	2.081	3.664	0.0005	0.2

with less effort than that involved in the method proposed in the author's paper. Professor Slade's Objection (*a*), under the heading, "Transformations of the Variate," can be met by the statement that his (Slade's) method requires the identical solution of the same type of cubic, as expressed in his own Equation (24). Moreover, it seems that his (Slade's) theoretical development, involving Equations (12) to (42), is unnecessarily unwieldy and considerably more complicated than the writer's (Fisher's) theory.

In "Example *b*" the author has quoted certain data from the writer's investigations and states "that the work required to fit these data by the present [Slade's] method is quite negligible compared to that which Fisher must do to obtain a result that is satisfactory for all practical purposes."

In the light of what has been said before in reference to the writer's own Equation (59), which is algebraically identical to, and easier to handle than,

the author's "partly bounded function," the foregoing and somewhat dogmatic assertion by Professor Slade becomes invalid in so far as the generating function, $\phi_0(x)$, is concerned.[36c]

In Table 13, the values of the symbols, b, c, and d, as used by Professor Slade,[36a] are compared with the corresponding values of the symbols, a (or u), m, and n, that enter into the formulas that the writer has used.[37] Referring to Column (3), Table 13, $\lambda_1(x) - u = a$. If the origin is taken at the mean, or at $\lambda_1(x)$, it is found that $\lambda_1(x) = 0$, or $a = -u$.

TABLE 13.—Comparison of Symbols—Slade and Fisher Methods

Slade		Fisher		Remarks
Symbol (1)	Value (2)	Symbol (3)	Value (4)	(5)
b	6.118	a (or $-u$)	6.118	Identical
c	1.681	n	0.4205	.....
c^2	2.826	$1:2n^2$	2.826	Identical
d	0.2131	m	1.5462	.....
$\log d$	—1.5462	$-m$	—1.5462	Identical

The logarithmically transformed variate (on the basis of age intervals of 1 yr) becomes,[37]

$$t = h(x) = [\log (x - 13.1) - 2.645] : 0.420 \quad \ldots\ldots\ldots (64)$$

with a generating function of the form:

$$\phi_0(x) = \frac{1}{0.420\sqrt{2\pi}} e^{-0.5[h(x)]^2} \quad \ldots\ldots\ldots\ldots\ldots (65)$$

in which, $h(x) = \left[\dfrac{\log (x - 13.1) - 2.645}{0.420}\right]$.

TABLE 14.—Computed Mid-Ordinates (See Table 5)

Ages (inclusive) (1)	Observed frequencies (2)	Computed mid-ordinates (3)	Ages (inclusive) (1)	Observed frequencies (2)	Computed mid-ordinates (3)	Ages (inclusive) (1)	Observed frequencies (2)	Computed mid-ordinates (3)	Ages (inclusive) (1)	Observed frequencies (2)	Computed mid-ordinates (3)
13 to 15	1	1	31 to 33	2 711	2 798	49 to 51	272	260	67 to 69	17	19
16 to 18	9	31	34 to 36	1 918	2 064	52 to 54	186	166	70 to 72	14	13
19 to 21	745	831	37 to 39	1 339	1 428	55 to 57	141	107	73 to 75	3	8
22 to 24	2 264	2 492	40 to 42	884	958	58 to 60	110	69	76 to 78	2	5
25 to 27	3 828	3 486	43 to 45	533	624	61 to 63	72	45	79 to 82	....	4
28 to 30	3 801	3 426	46 to 48	380	405	64 to 66	43	29	> 83	.....	6

The computed mid-ordinates, using the correctly computed generating function (Equation (65)), are listed in Table 14.

[36c] Correction (October, 1934, *Proceedings*): On page 1116, Lines 1, 2, and 3, "$t = 0.440$; $b = 6.118$; $c\sqrt{2} = 2.377$; and $d = 0.2131$". Numerical values as given by Professor Slade are erroneous.

[37] "The Mathematical Theory of Probabilities", by Arne Fisher, p. 258 ($m + \log 3 = 2.645$).

It is easily recognized that the generating function itself (or Professor Slade's "partly bounded function") is not powerful enough to represent the observations (which are not only skewed, but peaked or "top heavy"). Indeed this becomes evident if one computes the fourth-order semi-invariant, which serves to characterize the peaked character, or the flatness, of the curve. For this very reason it becomes necessary to resort to a more refined graduation by means of a Gram series of the form:

$$c_0 \phi_0(x) + c_3 \phi_3(x) + c_4 \phi_4(x) \ldots\ldots\ldots\ldots\ldots\ldots (66)$$

in which, $\phi_0(x)$ is the generating function as computed in Equation (65) and $\phi_3(x)$ and $\phi_4(x)$, its third and fourth-order derivatives. This further graduation may be carried out by least squares,[37] and results in the distribution shown in Table 15.

TABLE 15.—Frequency Distribution by Gram Series

Ages (inclusive) (1)	Computed frequencies (2)	Ages (inclusive) (1)	Computed frequencies (2)	Ages (inclusive) (1)	Computed frequencies (2)	Ages (inclusive) (1)	Computed frequencies (2)	Ages (inclusive) (1)	Computed frequencies (2)
13 to 15	0	28 to 30	3 707	43 to 45	519	58 to 60	88	73 to 75	21
16 to 18	72	31 to 33	2 899	46 to 48	342	61 to 63	66	76 to 78	16
19 to 21	696	34 to 36	1 996	49 to 51	233	64 to 66	50	79 to 82	13
22 to 24	2 369	37 to 39	1 284	52 to 54	163	67 to 69	38	>83	24
25 to 27	3 714	40 to 42	816	55 to 57	118	70 to 72	29	.....	....

Fig. 5, which demonstrates the superiority of the Gram series, affords a comparison between: (1) The original observations; (2) the computed frequencies based on a single generator, Equation (65); and, (3) the fre-

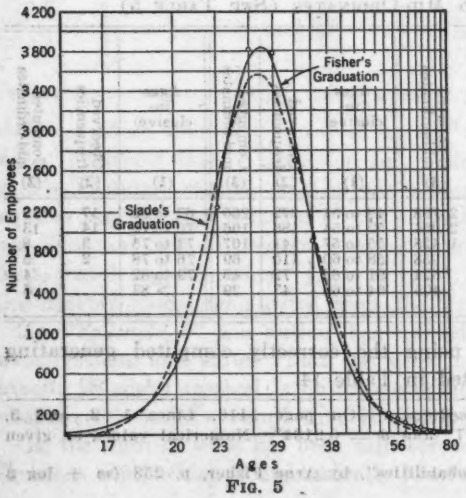

Fig. 5

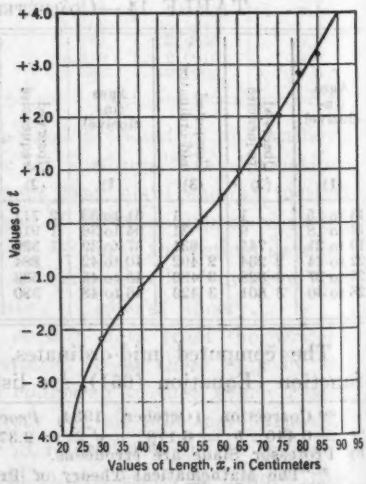

Fig. 6

quencies as computed from a Gram series. An application of the so-called test of "goodness of fit," as devised by Pearson and R. A. Fisher also shows that a single generating function does not suffice to describe the observations. The test for "goodness of fit" (which might better be termed a test for "poorness of fit") gives altogether too much weight, however, to the infrequent and rare observations toward the ends of the curve. A superior, although rather lengthy, test is offered through the so-called "error critique" by Thiele. This "error critique," which utilizes an ingenious system of weighting, also shows the superiority of the Gram series.

In reference to "Example *c*," giving forty-nine individual observations on rainfall, the writer recommends the following simple procedure in preference to the tedious method proposed by Professor Slade. If the Naperian logarithms of the forty-nine individual measures of rainfall are arranged in intervals of 0.1, the resulting frequency values are as shown in Table 16.

TABLE 16.—FREQUENCIES OF LOGARITHMS OF INCHES OF YEARLY RAINFALL

Logarithm of rainfall, z (1)	$F(z)$ (2)	Logarithm of rainfall, z (1)	$F(z)$ (2)	Logarithm of rainfall, z (1)	$F(z)$ (2)
3.4 to 3.5	2	3.8 to 3.9	11	4.1 to 4.2	.2
3.5 to 3.6	0	3.9 to 4.0	14	4.2 to 4.3	1
3.6 to 3.7	7	4.0 to 4.1	4	> 4.3	0
3.7 to 3.8	8	.....	..	.....	...

An elementary and rapid calculation yields the value, $\lambda_1(z) = M = 3.852$, for the mean, and $\sqrt{\lambda_2} = \sigma = 0.164$ as the value for the dispersion (or standard deviation). The simple normal exponential frequency curve, therefore, takes on the form of Equation (59) thus,

$$\phi_0(x) = \frac{49}{0.164\sqrt{2\pi}} e^{-0.5t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (67)$$

in which, $t = \left[\frac{\log x - 3.852}{0.164}\right]$. The resulting values, comparable to those in Table 7, are listed in Table 17.

TABLE 17.—COMPUTATION OF FREQUENCIES TO COMPARE WITH TABLE 7

x (1)	$\log x$ (2)	(2) − 3.852 (3)	(3) : 0.164 = t (4)	$I(t)$* (5)	49(5) (6)
30	3.401	−0.451	−2.750	0.00298	0.14
35	3.555	−0.297	−1.811	0.03507	1.72
40	3.689	−0.163	−0.988	0.16158	7.91
45	3.807	−0.045	−0.274	0.39204	19.20
50	3.912	0.060	0.366	0.64281	31.50
55	4.007	0.155	0.945	0.82766	40.55
60	4.093	0.241	1.470	0.92922	45.53
65	4.174	0.321	1.957	0.97482	47.76
80	4.382	0.530	3.232	0.99938	48.97

* Taken from "Table of Probability Integral," in "Statistical Methods," by C. B. Davenport, Table IV, p. 119, John Wiley & Sons, New York, 1904.

The fact that the logarithms of the measures of rainfall have been regarded as the observed quantities leads immediately to the Laplacean probability function, or normal error curve, without going through the more complicated calculations of the logarithmic transformation. This fact, however, leads one to question the undue stress that Professor Slade has put on the existence of upper and lower limits, situated within a finite range, instead of extending from minus infinity to plus infinity in the normal exponential error law. If one were to insist on definite upper and lower limits of the direct measures of rainfalls, why should one not also insist on definite limits, or a finite range of variation, for the logarithms of the measures?

The writer has referred briefly to Thiele's "pseudo-normal" variates, and has mentioned, as a special case, the transformation,

$$t = h(x) = \left[\log\left(\frac{x - x_0}{x_n - x}\right) - m \right] : n \quad \ldots\ldots\ldots\ldots (68)$$

in which, x_0 and x_n represent the upper and lower limits of the observations. This transformation can be shown to be identical to that proposed by Professor Slade in his "Most General Function," Equation (45a). To be sure the latter contains apparently five constants as against four in Equation (68), but the

TABLE 18.—Observations on the Lengths of Oat Stalks, in Centimeters

Values of x, in centimeters	$F(x)$	(x)	$\Sigma F(x)$	$\Sigma \phi(x)$	Anti-probability of $I(t)$	log (x)	log $(95-x)$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
<25	1	25	1	0.001	—3.090	1.398	1.845
25 to 30	12	30	13	0.013	—2.227	1.477	1.813
30 to 35	32	35	45	0.045	—1.696	1.544	1.778
35 to 40	62	40	107	0.106	—1.248	1.602	1.740
40 to 45	106	45	213	0.211	—0.803	1.653	1.699
45 to 50	138	50	351	0.348	—0.391	1.699	1.653
50 to 55	159	55	510	0.506	0.015	1.740	1.602
55 to 60	182	60	692	0.687	0.487	1.778	1.544
60 to 65	155	65	847	0.840	0.994	1.813	1.477
65 to 70	92	70	939	0.932	1.491	1.845	1.398
70 to 75	49	75	988	0.980	2.054	1.875	1.301
75 to 80	18	80	1 006	0.998	2.879	1.903	1.176
80 to 85	1	85	1 007	0.999	3.090	1.929	1.000
>85	1	∞	1 008	1.000	∞	1.954	0.699

Value of x, in centimeters	Column (7) minus Column (8)	Column (9) minus 0.129	Column (10) / 0.214 $= t$	$I(t)$	$\Delta I(t)$	1 008 times Column (13)	Observed $F(x)$
(1)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
<25	—0.477	—0.576	—2.692	0.0036	0.0036	3.6	1
25 to 30	—0.336	—0.465	—2.173	0.0149	0.0113	11.4	12
30 to 35	—0.234	—0.363	—1.696	0.0449	0.0300	30.2	32
35 to 40	—0.138	—0.267	—1.248	0.1060	0.0611	61.6	62
40 to 45	—0.046	—0.175	—0.818	0.2067	0.1007	101.5	106
45 to 50	0.046	—0.083	—0.388	0.3490	0.1423	143.4	138
50 to 55	0.138	0.009	0.042	0.5168	0.1678	169.1	159
55 to 60	0.234	0.105	0.491	0.6883	0.1715	172.9	182
60 to 65	0.336	0.207	0.967	0.8332	0.1449	146.1	155
65 to 70	0.447	0.318	1.486	0.9314	0.0982	99.0	92
70 to 75	0.574	0.445	2.079	0.9812	0.0498	50.2	49
75 to 80	0.727	0.598	2.794	0.9974	0.0162	16.3	18
80 to 85	0.929	0.800	3,738	0.9999	0.0025	2.5	2
>85	1.255	.....	.....	.....	.....	.....	...

fifth constant, corresponding to p in Equation (45*a*), is contained in the value of m. Moreover, in Professor Slade's Equation (45*d*) it appears that the constant, p, is expressed in terms of the other constants, so that after all there are only four constants.

"Example *d*," which Professor Slade has chosen to illustrate his "most general frequency function" is too fragmentary to be of much use. No observed statistical data are given, except in a rough graphical representation in Fig. 3 of the paper.

As an illustration of this function and also of the identically same type of transformation (Equation (68)) mentioned by Gram, the writer presents, in Table 18, Columns (1) and (2), some observations[38] by Dr. Giltay, of Wageningen, Holland, on lengths, in centimeters, of 1 008 oat stalks.

Column (1) shows the length, in centimeters, of the 1 008 stalks; Column (2) gives the frequencies, $F(x)$, in the intervals shown in Column (1); Column (4) contains the accumulated absolute frequencies; Column (5) lists the accumulated relative frequencies; and in Column (6) are the values of t in the probability integral, $I(t)$, corresponding to x in $\Sigma_{-\infty}^{x}\phi(x)$. This value of t is found by entering the table of the probability integral inversely which explains the heading for Column (5) as the anti-probability of $I(t)$.

The points, t, corresponding to x are plotted as small circles in Fig. 6. If a linear relation, $t = h(x) = (x - M) : \sigma$, is assumed, $t = (x - 54) : 11.8$, approximately. However, this gives a poor fit, because a glance at the curve, $t = h(x)$, shows that it has a pronounced point of inflection in its central region and tapers downward more rapidly than the straight line; and, likewise, it rises more rapidly upward than the straight line. Consequently, the normal error curve that implies a linear relation does not apply to the observations as they stand. The geometrical form of $t = h(x)$, moreover, leads to the assumption of finite upper and lower limits, or to postulate that,

$$t = h(x) = \left[\log \left(\frac{x - x_0}{x_n - x} \right) - m \right] : n \quad \ldots\ldots\ldots\ldots (69)$$

The next step, therefore, is to estimate the numerical values of these upper and lower limits, or to find an approximation of x_0 and x_n. It is possible to do so by several algebraic approximations; but this is a tedious process, and it is preferable to use graphical methods of which a great variety is available. A very rapid process has been suggested by the Netherlands mathematician, M. J. Van Uven.[38] An application of this process leads to the choice of x_0 as 0 and x_n as 95, so that,

$$t = h(x) = \left[\log \left(\frac{x}{95 - x} \right) - m \right] : n = k \log \left(\frac{x}{95 - x} \right) + c \ldots (70)$$

The calculation of the logarithms of the fractions, $x : (95 - x)$, is shown in Columns (7), (8), and (9), of Table 18.

[38] Koninklijke Akademie Van Wetenschappen te Amsterdam, Vol. 19, No. 4, pp. 533-545, from which the present example relating to oat stalks is taken.

The next step is now to find the values of the constants, k and c (or of m and n), in Equation (70). This is most easily done by the method of least squares. If the interval from $x = 35$ to $x = 70$ (in which the majority of the observations fall) is chosen, the anti-probability of $I(t)$ as shown in Column (5), Table 18, may be considered as an observed value of t that is to be expressed by the linear relation,

$$t = k \log \left(\frac{x}{95 - x} \right) + c \quad \ldots\ldots\ldots\ldots\ldots\ldots (71)$$

The coefficients of k, that is,

$$\log [x : (95 - x)] = a_x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (72)$$

are given in Column (8), Table 18, and 1.000 may be taken as a provisional or preliminary value of c. Hence, the following schemata of observation equations are obtained:

a_x	c	$o(x)$	a_x	c	$o(x)$
−0.234	1.000	−1.696	0.138	1.000	0.015
−0.138	1.000	−1.248	0.234	1.000	0.487
−0.046	1.000	−0.803	0.336	1.000	0.994
0.046	1.000	−0.391	0.447	1.000	1.491

From these observation equations one finds (in the notation of Gauss) that: $[aa] = \Sigma a^2 = 0.465$; $[ac] = \Sigma ac = 0.783$; $[ao] = \Sigma ao = 1.704$; $[cc] = \Sigma c^2 = 8.000$; and, $[co] = \Sigma co = -1.151$. Consequently, $0.465\,k - 0.783c = 1.704$; and, $0.783\,k + 8.000c = -1.151$; or $k = 4.6786$ and $c = -0.6018$. Therefore, $n = (4.6786)^{-1} = 0.214$, and $m = 0.6018 : 4.6786 = 0.129$; or t reduces to:

$$t = \left[\log \left(\frac{x}{95 - x} \right) - 0.129 \right] : 0.214 \quad \ldots\ldots\ldots\ldots (73)$$

The calculation of the graduated curve proceeds now (see Columns (10) to (15)) as a simple continuation of Columns (1) to (9) of Table 18. The detailed calculations are self-explanatory, and the final column (Column (14)) which gives the graduated curve, shows a reasonable agreement with the observed and ungraduated data in Column (2), or Column (15). (See Fig. 7.) The same curve type has been discussed by Van Uven, and it is of interest to compare his form with Equation (71). Van Uven writes t as,

$$t = \lambda \log \left[\left(\frac{x - x_o}{x_n - x} : \frac{x_m - x_o}{x_n - x_m} \right) \right] \quad \ldots\ldots\ldots\ldots\ldots (74)$$

in which, x_m, a fifth constant, analogous to the constant, p, in Professor Slade's Equation (45a), equals the value of the median of the observations. It is evident that the quantity, $\log \left(\frac{x_m - x_o}{x_n - x_m} \right)$, must fall very close to the value of m in Equation (69), because of the fact that t is assumed to be normally distributed, which implies that the mean value, m, coincides with

the value of t that corresponds to the median, x_m, of the observations. The median of the observed values of $F(x)$ falls close to 54.5 cm, so that $\log\left(\frac{x_m - x_0}{x_n - x_m}\right) = \log\left(\frac{54.5}{40.5}\right) = 0.1289$, which checks with the previously calculated value of $m = 0.129$.

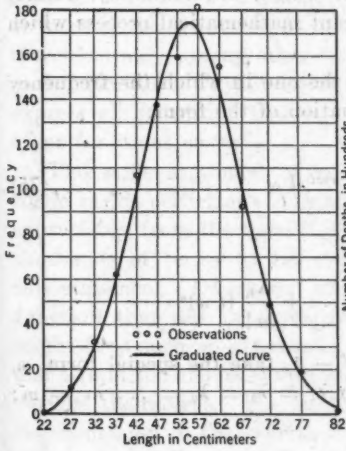

FIG. 7

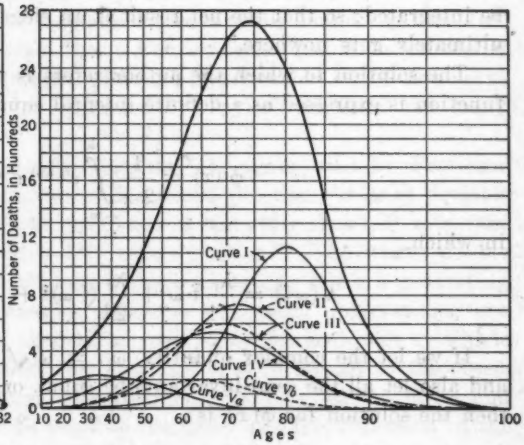

FIG. 8

The restriction of Van Uven to let the mean in the transformed curve equal the particular value of t that corresponds to the median of the observations is analogous, although not equivalent, to the introduction of the author's constant, p, to make the second moment of the final curve agree with that of the statistics, which at its best is nothing more than a kind of artificial *tour de force*.

In the opening paragraphs of his "Synopsis" Professor Slade refers to "a justification for the introduction of a new function in a field that literally teems with studies." Enough has probably been said in the preceding paragraphs to demonstrate that there is little, if any, newness in the formulas presented; they are, to all practical purposes, identical with those known from the old methods of logarithmic transformations of the variate.

All such methods go back to Thiele, who always took a critical attitude to the Gaussian law of error with its implied postulate of the arithmetic mean as the most probable value. As early as 1866, Thiele introduced the alternative postulate of the geometric mean as the most probable value, thus antedating McAllister, Fechner, Pearson, and other writers by many years.[39] Thiele's investigations led him to the obvious fact that if the geometric mean of the observations yields the most probable value, the arithmetic mean of the logarithms of the observations must yield the most probable value of the

[39] "The Mathematical Theory of Probabilities", by Arne Fisher, p. 288.

logarithms of the observations, so that if the logarithms of the observations are assumed to be normally distributed, a law of error can be found for the observations themseleves by a logarithmic transformation of the variate. Such a law must necessarily lead to the geometric mean of the observations as the most probable value.

In the text following Equation (7) Professor Slade states "in Thiele's solution, $\phi(x)$ is given as a definite integral which, unfortunately, cannot be integrated; so that the net result is an elegant mathematical process which ultimately gets nowhere."

The solution to which the author refers is the one in which the frequency function is expressed as a definite integral equation of the form:

$$\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{h(\omega i)} e^{-x\omega i} \, d\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots (75)$$

in which,

$$h(\omega i) = \frac{\lambda_1}{1!} i\omega + \frac{\lambda_2}{2!} (i\omega)^2 + \ldots + \frac{\lambda_n}{n!} (i\omega)^n$$

If we let the complex quantity, $\omega i = \omega\sqrt{-1}$, take the special form, ω, and also let all the semi-invariants be equal, or $\lambda_1 = \lambda_2 = \lambda_3 = \ldots \lambda_n = m$; then the solution for $\phi(x)$ is:

$$P(x) = \frac{e^{-m} m^x}{x!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (76)$$

which is the well-known error law of Poisson.[40]

Also, if all the semi-invariants of higher order than the second are found to vanish, the solution becomes,

$$\phi_0(x) = (2\pi\lambda_2)^{-0.5} e^{-h(x)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (77)$$

with $h(x) = (x - \lambda_1)^2 : 2\lambda_2$, or the normal error curve of Gauss.[41] In fact, one of the properties of the normal curve is that all its semi-invariants above the second order equal zero.

Thus, one finds that, not only the normal error law, but also Poisson's celebrated exponential, are contained as special cases in Thiele's general functions.

By resorting to an expansion in Gram's series and letting,

$$\phi(x) = c_0 \phi_0(x) + c_1 \phi_1(x) + c_2 \phi_2(x) + \ldots c_n \phi_n(x) \ldots\ldots (78)[42]$$

one can also solve Thiele's integral equation and use the solution for the numerical determination of the constants, c, in the Gram expansion, where $\phi_0(x)$ is the normal or Laplacean error curve in Equation (77).

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[40] "The Mathematical Theory of Probabilities," by Arne Fisher, p. 265.
[41] *Loc. cit.*, p. 198.
[42] *Loc. cit.*, pp. 209–210.

Of course, as Professor Slade points out, Thiele's general formula may contain other functions than frequency functions. To the writer this long and well-known fact actually enhances rather than detracts from the value of Thiele's work, for the more general a formula is, the greater is its applicability in mathematical analysis. Professor Slade, on the other hand, uses this circumstance as a criticism by stating that Thiele's solution is "too general" for what he somewhat ambiguously terms "ordinary purposes." Such an argument can scarcely be considered as mathematically sound. If accepted, it would imply, among other things, that a Fourier series should be considered as inferior to a Taylor power series, although the Fourier series is able to express all functions that can be expanded by Taylor's formula and many other functions as well.

The same argument of being "too general" is again used by Professor Slade in his criticism of Gram's series. He states also that "if $\phi(x)$ deviates appreciably from the normal curve, $\phi_0(x)$ (or is markedly skew), it may take thousands of terms to get even a fair representation." The writer doubts this statement. Of course, if one were foolish enough to select a generating function that did not even remotely resemble the rough observations, one might drop such an attitude of skepticism; but, as emphasized both by Thiele and Gram, the choice of the generating function is to be guided by the observations themselves. For instance, one should not always let the mean and the dispersion in the generator be equal to the mean and the dispersion of the observations. In greatly skewed or one-sided distributions, such as the classical example of *Ranunculus Bulbosus* by deVries,[42] the variate in the normal generator should not be taken as:

$$t = (x - 0.631) : 0.958 \ldots\ldots\ldots\ldots\ldots\ldots (79)$$

but as,

$$z = [\log (x - 4) - 0.044]\ 0.544 \ldots\ldots\ldots\ldots\ldots (80)$$

and Gram's series should be written as:

$$F(x) = c_0\,\phi_0(z) + c_1\,\phi_1(z) + c_2\,\phi_2(z) + \ldots\ldots\ldots\ldots (81)$$

Furthermore, the objection toward the possible occurrence of negative frequencies mentioned by Professor Slade can easily be overcome if, like Thiele and Charlier, one were to write the frequency function as,

$$F(x) = e^{\Sigma c_i \phi_i(x)} = e^{\Sigma c_i H_i(x)} \ldots\ldots\ldots\ldots\ldots\ldots (82)$$

(for $i = 0, 1, 2, 3, \ldots n$), in which, $H_i(x)$ represents the Hermite functions of the order, i. Taking the logarithms on both sides of Equation (82):

$$c_0 + c_1 H_1(x) + c_2 H_2(x) + \ldots c_n H_n(x) = \log F(x) \ldots\ldots\ldots (83)$$

[42] "The Mathematical Theory of Probabilities", by Arne Fisher, p. 226.

Multiplying both sides of Equation (83) with the ith derivative of the generator, $\phi_0(x)$, and integrating over the entire range:

$$c_i \int_{-\infty}^{\infty} H_i(x)\ \phi_i(x)\ dx = c_i\ (-1)^i\ i! = \int_{-\infty}^{\infty} \log F(x)\ \phi_i(x)\ dx$$

or,

$$c_i = \frac{(-1)^i}{i!} \int_{-\infty}^{\infty} \log F(x)\ \phi_i(x)\ dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (84)$$

for $i = 0, 1, 2, 3, \ldots .n$."[44]

The determination of the constants in the series through this method will never result in negative frequencies. In fact, the method outlined herein is one that can be highly recommended for curves of rainfall, especially since excellent tables of the Hermite functions are now available, progressing by intervals of 0.01.

Article 6, Section II of Professor Slade's paper is devoted to a discussion to prove the self-evident fact that as t approaches 0, or as the skewness vanishes, Equation (11) will converge toward the normal Laplacean curve as a limiting form. In order to show this, the author goes through a lengthy mathematical analysis of limits involving certain results originally presented by the French mathematician, L'Hospital. Of course, such complicated networks of formulas for limits become wholly superfluous under the methods advocated by Thiele because, if the skewness, or (what is the same thing) the third (and higher) order of semi-invariants vanish, an obvious corollary is derived from Thiele's theory that the resulting curve always is of Laplacean normal form.

A most interesting point that Professor Slade has very briefly touched upon is that of compound frequency curves (see "II—The Partly Bounded Function," Requirement (a)), a somewhat difficult problem from the standpoint of mathematics, but at the same time the one that is probably the most important in actual practice. Every frequency distribution observed in Nature is really a compound frequency curve, composed of several subsidiary error curves, either of normal or non-normal types.

In this connection, it is of inteest to point to the remark which the French mathematician, Poincare, made to his German colleague, Lindemann; "The mathematicians," Poincare said, "imagine that the law of errors is a fact of observation, and the observers fancy that it is a theorem demonstrated by mathematics." Neither one of these beliefs is true. As a matter of fact, the Russian mathematician, Tschuprov, in discussing whether or not it is possible to demonstrate by means of empirical methods the existence or non-existence of the exponential law of error, reached the conclusion that none of the numerous (and at times vast) collections, derived from precision and astronomical measurements, offers proof as to the actual existence of the postulated Gaussian error law. Tschuprov himself leaned toward the opinion that the observed statistical samples, especially the larger samples, tended

[44] "The Mathematical Theory of Probabilities," by Arne Fisher, pp. 201–202.

to discredit the actual existence in Nature of the celebrated law of error such as had been derived from purely mathematical speculations.

The Danish biologist, W. Johannsen, furthermore demonstrated by means of experimental research work in plant physiology that many of the biological variation curve types, discussed by Galton and Karl Pearson as normal error curves or skew curves, in reality were compound frequency curves, composed of two or more mathematical frequency curves. This important fact led Johannsen to the introduction in biological research of his celebrated "pure lines."

In other places the writer has made use of this idea of compound frequency distributions and "pure lines" in the construction of mortality tables from statistics, giving the deaths by sex, attained ages at death, and cause of death, but without knowing the therewith correlated number of living persons exposed to the risk of death.[45] In a sense this purely synthetic method constitutes a solution to the problem of determining the probable values of a series of proper fractions, where only the numerators are known, but where every one of the denominators are missing. Of course, it is impossible to solve this problem by mere mathematical or quantitative methods. The only approach toward a solution seems to be by means of partly quantitative and partly qualitative methods in the sense that the numerators, which show the observed statistics on death, do not only possess the quantitative attributes of attained ages at death, but also the qualitative attributes of the causes of death. These two elements in combination with certain biological hypotheses make at least an approximate solution possible by means of mathematical synthesis.

In Fig. 8 the result of such a construction is shown in relation to the mortality among Australian males for the period 1907–1915. The compound curve represents the well-known d_x column, or "curve of death," as found in all first-class mortality tables; the subsidiary causal or biological components give the death curves from specified groups of causes of death.

As pointed out by Dr. Raymond Pearl, the synthesis as proposed by the writer moves along entirely different lines than the earlier frequency curve study by Karl Pearson in his interesting volume on "The Chances of Death," for which Dr. Pearl states that "actually there is no slightest reason to suppose that it [Pearson's method] represents any *biological* reality."[46]

Moreover, the synthesis is of so general a nature that it may equally well be applied to engineering problems. Mr. Bassett Jones,[47] for example, has made use of it in the design and construction of high-speed elevators, and it seems to be equally well adapted for the study of rainfall by taking into consideration the therewith related geographical, climatic, and geological elements.

A question which is of the utmost importance to engineers and observers, who usually deal with limited samples, is that of the mean (or probable) errors

[45] "Frequency Curves," by Arne Fisher, Macmillan Co., N. Y., 1922, and also article in 1925 *Skandinavisk Aktuarietidskrift.*

[46] "The Biology of Death", by Raymond Pearl, Philadelphia, Pa., 1922, p. 101.

[47] "Power Calculations for Elevators by the Method of Probabilities', by Bassett Jones, *General Electric Review*, 1930, pp. 545–555 and 650–653.

of the constants or parameters. Earlier formulas for probable errors of the constants in certain types of frequency curves derived by graphical methods[48] are defective because their derivation rests on the implied assumption that the constants are mutually free or uncorrelated. This implication is faulty. The constants in such curves are correlated in the sense that they all depend on the lower (or upper) limits of the variate; in fact, the constants are mathematical functions of the limits, which in the graphical process are chosen subjectively or by the mere aid of the eye, as it were. The least bit of change in such limits will influence the other constants.

The obstacle of correlation between the constants exists also in the logarithmic transformation with a lower limit (or mathematical zero) in the generator as defined by Equation (57). The writer as well as Mr. Sture Nydell have succeeded, however, in overcoming this obstacle and have been able to give the following approximate formulas for the mean (or standard) errors of u (or a), m, and n:

$$E_m\,(u) = (1 + t)\,t^{-1}\,[(8 + 106t + 333t^2 + 479t^3) : 12s]^{0.5}$$

$$E_m\,(m) = [(4t^{-1} + 54 + 306t + 895t^2 + 1\,598t^3 + 1\,753t^4) : 6s]^{0.5}$$

$$E_m\,(n) = n^{-1}\,[t\,(8 + 106t + 332t^2 + 479t^3) : 12s]^{0.5}$$

in which E = the mean error; s, the number of observations (or size of sample); and $t = \lambda_2 : u^2$.

In Professor Slade's "Example *a*," one finds $\lambda_2 = 1.442$; $u = 1.948$; $t = 0.3800$; $m = 0.1823$; $n = 0.3680$; and $s = 251$, so that $E_m\,(u) = 0.4938$; $E_m\,(m) = 0.5367$; and $E_m\,(n) = 0.1474$, which shows that the value of the logarithmic mean, m, is decidedly uncertain because its mean error is approximately three times the value of m itself.

[48] See, for example, *Transactions*, Am. Soc. C. E., Vol. 91 (1927), p. 1.

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DISCUSSIONS

UNDERGROUND CORROSION

Discussion

BY MESSRS. THOMAS F. WOLFE, AND JOHN R. BAYLIS

THOMAS F. WOLFE,[13] M. AM. SOC. C. E. (by letter).[13a]—A concise description of the corrosion study being conducted at the National Bureau of Standards, is presented in this paper. In spite of the fact that metal pipes have been used underground in the United States for considerably more than a century, the investigation of the corrosive effects of soil had not been given the attention it deserved until investigators at the Bureau began their studies, which were based on a long-time plan and covered a number of different types of soil. In some of the tests conducted prior to this development, changes in management or in the personnel resulted either in discontinuing the test or in the loss of some of the data that would have been useful in interpreting the results. The tests at the Bureau of Standards have been based on a definite plan and from the very beginning have been under the direction of one man continuously thereafter.

The impetus responsible for the intensive study of soil corrosion was probably furnished by the fact that oil and gas lines installed in the early days of cross-country transmission were beginning to show signs of deterioration. Many of these lines were laid in soils that were more corrosive than soils in other sections of the country where pipes of considerably greater age were still giving service. These older pipes as a rule were cast-iron water mains and gas mains and had wall thicknesses in excess of that in the pipe used for gas and oil transmission lines.

The original idea behind the tests at the Bureau was not to develop information concerning relative merits of pipe materials when exposed to soil corrosion, but rather to determine the corrosive qualities of soils. However, the use of a number of types of ferrous[13b] pipe specimens in the work has led some engineers to use the data for the comparison of materials. Although the con-

NOTE.—The paper by K. H. Logan, Esq., was published in March, 1935, *Proceedings*. This discussion is printed in *Proceedings*, in order that the views expressed may be brought before all members for further discussion.

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[13a] Received by the Secretary February 6, 1935.

[13b] Corrections to be made in the paper before it is finally published in *Transactions:* March, 1935, *Proceedings*, p. 320, in caption to Fig. 1, change "Steel Pipe" to "Iron Pipe," and "(*d*)" to "(*f*)" and "(*f*)" to "(*d*)." In the captions to Figs. 1, 2, and 3 (pp. 320–321), change "Eight Years" to "Ten Years," and change "(*a*)" to "(*c*)" and "(*c*)" to "(*a*)."

tinuation of the tests described by Mr. Logan, at some future time, may warrant drawing conclusions as to relative lives of materials under similar soil conditions, it would seem wise at this time to reserve judgment on this phase of the problem until more data are available. It cannot be stressed too strongly that this work is really pioneer work and, although of considerable value, great care should be taken in using data that are necessarily based on short-time testing.

One of the dangers to be avoided in the use of these soil corrosion data is to attempt at this time to develop any hard and fast rule regarding pipeline depreciation. The best guide as to what depreciation actually occurs is to examine specimens removed from the property under consideration. The attempt to correlate any given soil with one of the soils presented in the paper, without having an intimate knowledge of soils, may lead one far astray. Even more serious error would result from attempting to predict depreciation according to the geographic location of pipe; for instance, one of the most corrosive soils under test is peat. The only test specimens buried in this particular soil in the original tests happens to be in Milwaukee, Wis. However, very few of the pipe lines in service in Milwaukee are actually laid in peat soil; consequently, any one examining the data and predicting a life for Milwaukee pipe based on the specimens in Milwaukee would be very much in error.

In addition to valuing pipe lines the data may finally come into use in the selection of materials for underground pipe construction. Under some conditions longevity is of much more importance than under other conditions. For instance, a gathering line in an oil field that has a limited life need last no longer than the field. On the other hand, a water line in a city street would properly be considered a permanent necessity. The economical angle varies also with location. Maintenance, reconditioning, or re-laying of cross-country lines at comparatively short intervals may be satisfactory from the standpoint of the operator of the lines, because of the fact that such lines are in open country and the excavation costs are relatively small. In the case of lines in city streets, conditions are reversed. The cost of the work is high because of the existence of pavements. As a rule the lines can only be shut down at great inconvenience to the consumers. In addition to the cost feature, the effect on public relations of repeatedly opening streets is very bad. The net result is that a greater expenditure to provide additional life is warranted for water and gas main construction within cities.

When further experience becomes available, the use of soil surveys to determine the amount and type of protective coating necessary may develop a regular procedure from which tangible results may be expected. It is doubtful if soil surveys within cities will be particularly useful. In many cities street grades have been changed by filling with cinders, ashes, slag, and various types of waste material of a more or less corrosive nature. In other cities, the construction of pavements and the installation of sewers, conduits, and other underground utilities may have disturbed the soil so that it no longer resembles the natural soil of that particular region.

The tests of coatings discussed in the paper have not been in process long enough to make it possible to draw any conclusions as to which type of coating would be suitable for all conditions. The general conclusion can be drawn, however, that it is difficult to predict what additional life may be expected when any of the ordinary coatings are used. Coatings which, in themselves, might appear satisfactory are often disturbed during installation, and although the coating that remains on the pipe may be effective, much of it may be removed, with the net result that corrosion may still continue its destructive work. The fact that the tests have shown that the rate of corrosion decreases with time would indicate that one of the best means of prolonging the life of a pipe is to provide additional wall thickness. The advantage of additional wall thickness over a coating is that the thicker pipe needs no special handling to avoid loss of its added protective qualities.

When one considers that hundreds of millions of dollars of pipe lines are now in service underground, one cannot help but be impressed by the necessity for developing information on soil corrosion. It is to be hoped that the Bureau of Standards will be able to continue its investigations and that from time to time new specimens and new soils may be brought under the test to the end that, at some future time, soil corrosion data will be so extensive that they can be used by the Engineering Profession both in valuation and design work to the same extent as other scientific data.

JOHN R. BAYLIS,[14] ASSOC. M. AM. SOC. C. E. (by letter).[14a]—A brief but important factual summary of the principal findings in the field of underground storage, is contained in this paper. The writer is in full accord with Mr. Logan's interpretation of the facts as revealed by the soil corrosion study on practically all phases of the work. Certainly, this problem is now much better understood than before the work at the National Bureau of Standards was started.

The main comments that can be made by the writer, except to praise the work, is to urge that it be continued and that it be extended to include varying the conditions for the chief types of soils in which pipe are now being used. Mr. Logan has shown that there is marked similarity in the corrosion of the various ferrous metals buried in the same locality. In the localities where the samples of pipe are now buried, it is likely that such factors as moisture, temperature, penetration of oxygen into the soil, and acidity of the soil, are the same for samples buried at a certain point, although they vary for different localities. No one knows definitely the effect of varying the moisture and other factors that influence the rate of corrosion.

Another series of tests might well be undertaken with provisions for varying most if not all of the factors influencing the corrosion. Part of the soil should be covered to represent a pavement. Other parts should be subjected to artificial rainfall. An impervious stratum should be constructed under another section of the soil so that some samples may be permanently under water in the soil, others under water part of the time and out of water part

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of the time, and other samples out of the water all the time, except for the moisture received from rains or from capillary attraction. These are a few of the variables that should be studied.

It is likely that soils have been selected already which vary widely in moisture content, temperature, acidity, etc., but there are so many factors influencing the results that the effect of each factor for each type of soil should be studied. This implies costly experimentation, but the enormous value of pipe now buried in the ground and gradually depreciating in value, due to corrosion, justifies the expenditure of a very substantial sum to obtain full information on the subject.